Attitude State Estimation with Multi-Rate Measurements for Almost Global Attitude Feedback Tracking

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A state estimation scheme that does not depend on the statistical distribution of bounded measurement noise is presented. This scheme is used to provide state estimates for feedback in an attitude tracking control scheme that exhibits almost global asymptotically stable tracking of a desired attitude trajectory with perfect state measurements. The control and estimation schemes use the global, unique representation of rigid body attitude provided by rotation matrices. Attitude and angular velocity state estimate updates are obtained from discrete multi-rate measurements using a deterministic filtering scheme. Propagation of discrete state estimates is carried out with a Lie group variational integrator, which preserves the orthogonality of rotation matrices during numerical propagation without reprojection. This integrator is also used to numerically simulate the feedback system. The performance of this attitude tracking control scheme is then compared with that of a recently reported quaternion observer-based continuous feedback attitude tracking scheme. This quaternion-based attitude tracking scheme is shown to exhibit unstable, unwinding behavior. Numerical

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simulation results for both these feedback tracking schemes are obtained for a satellite in circular Earth orbit tracking an oscillating angular velocity spin maneuver. These numerical results are then compared for control effort and time taken for the tracking errors to converge to the desired trajectory.

I. Introduction

Attitude estimation and control of rigid bodies in gravity have applications to motion control of spacecraft, aircraft, underwater vehicles, and mobile robots. Rigid body attitude control has been a benchmark problem in nonlinear control, studied under various assumptions and scenarios in the past [1–7]. For feedback attitude tracking, attitude and angular velocity estimates are usually constructed from discrete-time measurements in practice. Attitude measurements are obtained indirectly through direction or angle measurements, usually at a slower rate than angular velocity measurements which are directly obtained from rate gyro. Therefore, several attitude estimation schemes estimate only the attitude and sometimes the rate gyro bias from these measurements [8–12]. Attitude estimates are then propagated between measurements using a filter and measured angular velocities, which are often assumed to be error-free; these estimates are then used for feedback control. This work presents an attitude and angular velocity estimation scheme that constructs state estimates from multi-rate measurements with bounded errors, and then uses this estimation scheme in conjunction with an almost global feedback attitude tracking control scheme that has appeared previously.

Most of the prior research on attitude control and estimation has been carried out using minimal (three-coordinate) or quaternion representations of the attitude. Any minimal representation necessitates a local analysis since such representations have kinematic singularities and cannot represent attitude globally. The quaternion representation is ambiguous, since for every possible attitude there are two sets of unit quaternion representations. Since unit quaternions distinguish between principal angle rotations of 0 and $2\pi$, this leads to instability of continuous feedback controllers and observers based on the quaternion representation of attitude. In the case of continuous quaternion feedback
controllers, the spacecraft may be needlessly rotated from principal angle $2\pi$ to 0, resulting in lack of Lyapunov stability of the desired equilibrium or trajectory. This occurs even though the desired state or state trajectory may be attractive with a large domain of attraction. Asymptotic stability, however, requires both Lyapunov stability and attractivity of the desired state(s) in the feedback system [13]. This instability of the feedback attitude dynamics with continuous quaternion feedback has been termed the unwinding phenomenon [2]. To avoid unwinding, discontinuous quaternion-based controllers have been used [14, 15]. However, discontinuous dynamics entail special difficulties [16], and may lead to chattering in the vicinity of a discontinuity, especially in the presence of sensor noise or disturbance inputs. Discontinuous controllers are also difficult to implement using actuators like reaction wheels, control moment gyros and magnetic torquerods that are commonly used for attitude control in spacecraft. It is thus important to determine which closed-loop properties can be achieved under continuous feedback control based on state estimates constructed from noisy measurements.

A global state estimation scheme that is applicable to arbitrary, uncontrolled rigid body rotational motion appeared in [17, 18]. This estimator uses the global and unique representation of attitude provided by orthogonal (rotation) matrices, like the observers for mechanical systems on Lie groups reported in [19, 20]. However, unlike these observers, the estimator in this paper takes into account discrete-time state measurements obtained from sensors operating at different sample rates and having bounded sensor noise. While the estimation scheme in [18] applied to a rigid body in gravity with no non-conservative torques and full state measurements, the estimation scheme in this paper is generalized to apply to controlled rigid body rotation in gravity, with a control torque obtained from continuous feedback of state estimates provided by the estimator. It is used here in conjunction with a feedback control torque obtained from an attitude tracking scheme with almost global asymptotic stability of the desired attitude and angular velocity trajectory [21–24]. Almost global asymptotic stability of a desired state trajectory means that its domain of attraction is the entire state space minus a subset of measure zero [22, 25]. This control scheme also does not exhibit unwinding, as shown in [25]. An adaptive version of this control scheme was shown to be robust to external disturbances in the absence of knowledge of the inertia [26]. For the measurement model,
it is assumed that attitude is measured indirectly through direction vector measurements, using the
attitude determination scheme in [17, 18]. Angular velocity measurements are obtained directly at
a faster rate than attitude measurements; therefore, a majority of the measurements are partial
state measurements consisting of angular velocity measurements only. No assumptions are made
for the particular noise distribution or stochastic properties of the sensor measurements; rather,
all measurements are considered to have bounded errors. These errors are bounded by ellipsoidal
bounds in the linearized state space [18]. Such deterministic estimation schemes have been shown
to be robust to the statistics of certain types of measurement noise [27, 28].

The dynamics model, which can describe arbitrary rigid body attitude motion without am-
biguities or discontinuities, is used for propagation of state estimates and uncertainty bounds on
these estimates between discrete-time attitude and angular velocity measurements. This estimation
scheme uses a filtering (update) algorithm based on intersections of the bounded sets of measurement
errors and dynamically propagated state estimate errors. Since these error bounds are deterministic,
changes in the stochastic properties of these errors over time and for different sensors do no adversely
affect the performance. The use of this estimation scheme for full state measurements in feedback
attitude tracking was reported in [29]. In [30], this scheme is extended to the practical case of multi-
rate measurements when attitude measurements are obtained at a slower rate than angular velocity
measurements. In this paper, the complete derivation of this estimator for multi-rate attitude and
angular velocity measurements is provided, along with its use with an almost global attitude and
angular velocity feedback tracking scheme. The performance of this estimator-based feedback atti-
tude tracking scheme is compared with an observer-based continuous quaternion feedback attitude
tracking control scheme reported in [31].

For numerical propagation of state estimates in our estimation scheme and for numerical simu-
lations, the equations of motion are discretized using the principles of discrete variational mechan-
ics [32]. Since a global state representation is required for arbitrary rigid body motion, general
purpose integrators like the Runge-Kutta schemes, which rely on use of local coordinates to de-
scribe the motion, are not applicable. Instead, a Lie group variational integrator, that gives us the
discrete equations of motion, is used. The idea behind variational integrators is to discretize the
variational principles of mechanics; in this case, the Lagrange-d’Alembert principle for a system with non-conservative forcing [32]. Use of this integration scheme ensures that the structure of the group of rigid body rotations is maintained, and that angular momentum is numerically conserved in the absence of control and external torques.

This paper is organized as follows. Section II gives the dynamics model of a rigid body in gravity with a feedback control scheme that converges almost globally to a prescribed bounded trajectory on the state space. Discrete equations of motion for the system, obtained as a Lie group variational integrator, are presented in section III. This section also presents the linearized discrete equations of motion. In section IV, the estimation scheme using discrete-time multi-rate measurements, and intersections of uncertainty ellipsoids to filter these measurements, is introduced. Section V presents the combination of the tracking control scheme and the estimation scheme. Section VI presents a quaternion observer-based feedback attitude tracking scheme that was reported in [31]. In section VII, the performance of the combined estimation and attitude feedback tracking schemes is compared to that of the quaternion observer-based feedback tracking scheme of section VI. This comparison is carried out by applying both these compensator schemes to tracking a desired state trajectory for the dynamics model of an initially tumbling satellite in circular Earth orbit. Numerical simulation results for both these tracking schemes are shown for this particular tracking task. Section VIII concludes this paper by summarizing the results and discussing applications of the schemes presented in this paper.

II. Attitude Dynamics and Trajectory Tracking

A. Equations of Motion for Attitude Dynamics

The rigid body attitude kinematics and dynamics are described here using rotation matrices for attitude representation. This dynamics model is later discretized to propagate state estimates between attitude and angular velocity measurements. Consider the attitude dynamics of a rigid body in the presence of a control moment and a potential that is dependent only on the attitude. The configuration space for attitude dynamics is the special orthogonal group SO(3) of $3 \times 3$ real matrices with determinant $+1$, also known as the rotation group. The space of $3 \times 3$ real skew-
symmetric matrices, which is the Lie algebra of $\text{SO}(3)$, is denoted $\mathfrak{so}(3)$. Define the vector space isomorphism $\cdot^\times : \mathbb{R}^3 \rightarrow \mathfrak{so}(3)$, by

$$
\Omega^\times = \begin{bmatrix}
\Omega_1 \\
\Omega_2 \\
\Omega_3
\end{bmatrix}^\times = \begin{bmatrix}
0 & -\Omega_3 & \Omega_2 \\
\Omega_3 & 0 & -\Omega_1 \\
-\Omega_2 & \Omega_1 & 0
\end{bmatrix},
$$

which identifies $\mathfrak{so}(3)$ with $\mathbb{R}^3$. For the remainder of this paper, the state of the system is described by $(R, \Omega) \in \text{SO}(3) \times \mathbb{R}^3$ where $R$ is the rotation matrix from the body-fixed coordinate frame to the inertial coordinate frame, and $\Omega$ is the angular velocity expressed in the body-fixed frame. The attitude kinematics is expressed by Poisson’s equation

$$
\dot{R} = R\Omega^\times.
$$

Note that $(\cdot)^\times$ is also the cross-product operator in $\mathbb{R}^3$, i.e., $x^\times y = x \times y$. The inverse of the map $(\cdot)^\times : \mathbb{R}^3 \rightarrow \mathfrak{so}(3)$ is denoted by $(\cdot)^\vee : \mathfrak{so}(3) \rightarrow \mathbb{R}^3$.

Let $U : \text{SO}(3) \rightarrow \mathbb{R}$ be a gravity potential dependent only on the attitude; this could be due to uniform gravity or central gravity (for example, a satellite in circular Keplerian orbit). The attitude dynamics is given by

$$
J\ddot{\Omega} = J\Omega \times \Omega + M_g(R) + \tau,
$$

where $\tau$ is the control torque. $M_g(R)$ is the moment due to the gravity potential $U(R)$, and is given by

$$
M_g^\times(R) = \left(\frac{\partial U}{\partial R}\right)^T R - R^T \frac{\partial U}{\partial R},
$$

where the partial derivative $\frac{\partial U}{\partial R} \in \mathbb{R}^{3 \times 3}$ is defined such that $\left(\frac{\partial U}{\partial R}\right)_{ij} = \frac{\partial U}{\partial R_{ij}}$. This dynamics model, with an additional disturbance moment due to atmospheric drag, was considered for a disturbance rejecting feedback tracking scheme [22]. A similar dynamics model, without control torques, was also used to study the attitude estimation problem [17, 18].

B. Trajectory Tracking

The attitude and angular velocity trajectory tracking control scheme that is used in conjunction with our estimation scheme is briefly described here. Since this scheme has appeared in prior related
work \cite{22, 26, 29}, details of its derivation or proof of its stability properties are not provided here.

Let the desired state trajectory be given by the desired attitude $R_d(t)$ and the desired angular velocity $\Omega_d(t)$, for some interval of time $t \in [0,T]$, where $T > 0$. Further, let $\Omega_d(t)$ and $\dot{\Omega}_d(t)$ be bounded and continuous during this time interval, such that the rate of change of the desired attitude satisfies

$$\dot{R}_d(t) = R_d(t)\Omega_d(t)^\times. \quad (3)$$

Now define the attitude and angular velocity tracking errors as follows

$$Q(t) := R^T_d(t)R(t), \quad \omega(t) := \Omega(t) - Q^T(t)\Omega_d(t). \quad (4)$$

Thus $R_d(t)$ is a $C^2$ trajectory on $SO(3)$, but otherwise arbitrary.

These definitions along with equation (3) lead to the attitude error kinematics equation

$$\dot{Q} = Q\omega^\times, \quad (5)$$

where the identity $(F\eta)^\times = F\eta^\times F^T$ holds for all $F \in SO(3)$ and all $\eta \in \mathbb{R}^3$. The angular velocity error dynamics is determined by

$$J\dot{\omega} = J(\omega^\times Q^T\Omega_d - Q^T\dot{\Omega}_d) - (\omega + Q^T\Omega_d)^\times J(\omega + Q^T\Omega_d) + M_g(R_dQ) + \tau. \quad (6)$$

The trajectory tracking error kinematics (5) and dynamics (6) depend on $Q, \omega, \Omega_d, \dot{\Omega}_d$, the moment due to the potential $M_g = M_g(R_dQ)$, and the control moment $\tau$.

The control torque is given by the following control law \cite{22}

$$\tau = -L\omega + JQ^T\dot{\Omega}_d + (Q^T\Omega_d)^\times JQ^T\Omega_d$$

$$+ \Phi'(\text{tr}(K - KQ)) \sum_{j=1}^3 k_j e_j^\times Q^T e_j - M_g(R_dQ). \quad (7)$$

Here $L = L^T$ is a positive definite matrix, $K = \text{diag}(k_1, k_2, k_3)$ with $k_3 > k_2 > k_1 > 0$, and $e_i$ is the $i$th canonical unit vector. The function $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ is a $C^2$ function that satisfies $\Phi(0) = 0$ and $\Phi'(x) > 0$ for all $x \in \mathbb{R}^+$. Furthermore, let $\Phi'(-) \leq \alpha(\cdot)$ where $\alpha(\cdot)$ is a Class-K function \cite{13}. This ensures that $\Phi(\text{tr}(K - KQ))$ is a Morse function on $SO(3)$ whose critical points are non-degenerate and hence isolated, according to the Morse lemma \cite{33}. It has been shown in previous publications \cite{22, 26, 29} that $(Q, \omega) = (I, 0)$ is an almost globally asymptotically stable equilibrium.
of the error dynamics (5)-(6) with control torque (7). Therefore, the desired attitude trajectory is tracked from almost all initial states in the state space. Ref. [24] generalizes this tracking control scheme to rigid body dynamics on the group of translational and rotational motions, $SE(3)$.

III. Discretization of the Dynamics for Numerical State Propagation

In this section, a discrete time model for the feedback dynamics of the rigid body described in the previous two sections, is developed using a discrete variational approach. The discrete model obtained is a Lie group variational integrator. However, this discrete model includes the non-conservative feedback control torque obtained from the robust almost global tracking control scheme given in the earlier section. This Lie group variational integrator for the controlled rigid body is derived using the discrete Lagrange-d’Alembert principle. It preserves both the mechanical structure as well as the geometry of the state space of attitude motion.

A. Discretizing the Kinematic Equations

Let $f_k$ denote the discrete approximation to a continuous time-varying quantity $f$ at time $t_k = t(k)$ for integer $k$. Let $h \neq 0$ be a fixed time step size, i.e., $h = t_{k+1} - t_k$. Integrating the kinematics equation $\dot{R} = R\Omega^\times$ with the assumption of constant angular velocity in the time interval $[t_k, t_{k+1}]$, one obtains

$$R_{k+1} = R_k F_k,$$

where $F_k = \exp(f_k^\times) \in SO(3)$ is a first order approximation to $\exp(h\Omega_k^\times)$ given by

$$F_k \approx \exp(h\Omega_k^\times) \approx I + h\Omega_k^\times.$$  \hspace{1cm} (8)

By ensuring that $F_k \in SO(3)$, it is guaranteed that $R_k$ evolves on $SO(3)$. Defining the modified inertia matrix $\mathcal{J} := \frac{1}{2} \text{trace}[J^T[I - J]$, where $I$ is the $3 \times 3$ identity matrix, it can be verified that for any $\Omega \in \mathbb{R}^3$ the following identities hold:

$$(J\Omega)^\times = \Omega^\times \mathcal{J} + \mathcal{J} \Omega^\times, \hspace{1cm} (9)$$

$$(J\Omega, \Omega) = \langle \mathcal{J}\Omega^\times, \Omega^\times \rangle , \hspace{1cm} (10)$$
where the inner product on the space of \( n \times m \) real-valued matrices is defined by

\[
\langle A, B \rangle := \text{tr}(A^T B).
\]

A defining relation between \( F_k \) and \( \Omega_k \) is formed by inserting the approximation (8) into the identity (9):

\[
(J\Omega_k)^\times = \Omega_k^\times J - J(\Omega_k^\times)^T \approx \frac{1}{\hbar} \left((F_k - I)J - J(F_k^T - I)\right)
\]

\[
= \frac{1}{\hbar} (F_k J - J F_k^T).
\]

Hence the discrete counterpart of the kinematics equations (1) becomes

\[
(J\Omega_k)^\times = \frac{1}{\hbar} (F_k J - J F_k^T),
\]

\[
R_{k+1} = R_k F_k,
\]

where (11) is first solved for \( F_k \), and then (12) is used to obtain \( R_{k+1} \).

B. Discretizing the Dynamics Equations via the Discrete Lagrange-d’Alembert Principle

A variational integrator is obtained by discretizing the Hamilton’s principle or the Lagrange-d’Alembert principle (in the presence of non-conservative forces and torques), rather than the continuous equations of motion as is the case for a generic Runge-Kutta method. For a thorough exposition of the discretization of Hamilton’s principle and Lagrange-d’Alembert principle, the reader is directed to ref. [34]. The discrete version of the Lagrange-d’Alembert principle using generalized coordinates is also given in ref. [32]. However, since \( R \in \text{SO}(3) \) is not a generalized (local) coordinate description of the attitude configuration, the discretization given in ref. [32] cannot be used.

Let \( U : \text{SO}(3) \rightarrow \mathbb{R} \) denote the potential energy map. The Lagrangian for the motion of the rigid body in \( \text{SO}(3) \) described in Section II is given by

\[
\mathcal{L}(R, \Omega) = \frac{1}{2} \langle \mathcal{J}\Omega^\times, \Omega^\times \rangle - U(R).
\]

The continuous Lagrange-d’Alembert principle giving system (2) states that

\[
\delta \int_0^T \mathcal{L}(R, \Omega) dt + \int_0^T (\tau^T \Sigma) dt = 0,
\]

\( \delta \int_0^T \mathcal{L}(R, \Omega) dt + \int_0^T (\tau^T \Sigma) dt = 0 \)
where $\tau$ denote the external moment in the body frame. As given in ref. [35], $\Sigma$ is obtained from the variation $\delta R$ by $\delta R = R\Sigma^\times$. This leads to the variation in $\Omega$ being expressed by $\delta\Omega^\times = \dot{\Sigma}^\times + [\Omega^\times, \Sigma^\times]$, where $\Sigma^\times \in \mathfrak{so}(3)$ and $[\cdot,\cdot]$ denotes the matrix commutator.

A discrete Lagrangian $\mathcal{L}_d$ approximates a segment of the action integral

$$\mathcal{L}_d|_k \approx \int_{t_k}^{t_{k+1}} \mathcal{L}(R,\Omega)dt,$$

Similarly, $\mathcal{F}$ approximates a segment of the virtual work integral

$$\mathcal{F}|_k \approx \int_{t_k}^{t_{k+1}} (\tau^T \Sigma) dt.$$

The discrete dynamics, implementable as a variational integrator, is then obtained from the discrete Lagrange-d’Alembert principle

$$\delta \sum_{k=0}^{N-1} \mathcal{L}_d|_k + \sum_{k=0}^{N-1} \mathcal{F}|_k = 0,$$

where $\Sigma_0 = \Sigma_N = 0$. The discrete Lagrangian is defined by

$$\mathcal{L}_d|_k := \frac{h}{2} \left( \frac{1}{h} (F_k - I), \frac{1}{h} (F_k - I) \right) - \frac{h}{2} \left( U(R_k) + U(R_{k+1}) \right).$$

The discrete virtual work $\mathcal{F}$ is defined by

$$\mathcal{F}|_k := h\tau^T_k \Sigma_{k+1} = \frac{h}{2} \langle \tau^\times_k, \Sigma^\times_{k+1} \rangle,$$

where

$$\tau_k := \frac{1}{2} \left( \tau(R_k,\Omega_k,t_k) + \tau(R_{k+1},\Omega_k,t_{k+1}) \right).$$

Variations of $F_k$ are obtained from the variations in $R_k$ as $\delta F_k = F_k \Sigma^\times_{k+1} - \Sigma^\times_k F_k$. For ease of notation, let us define $U_k = U(R_k)$. Therefore, the first term in the discrete Lagrange-d’Alembert principle (14) can be computed as

$$\delta \sum_{k=0}^{N-1} \mathcal{L}_d|_k = \frac{h}{2} \sum_{k=0}^{N-1} \left( \frac{1}{h} (F_k - I), \frac{1}{h} (F_k - I) \right) - \frac{h}{2} \left( U(R_k) + U(R_{k+1}) \right).$$

Now using (11), the facts that symmetric and skew-symmetric matrices are orthogonal in the trace inner product and $\Sigma_0 = \Sigma_N = 0$, this expression can be rewritten as

$$\delta \sum_{k=0}^{N-1} \mathcal{L}_d|_k = \frac{h}{2} \sum_{k=1}^{N-1} \langle \Sigma^\times_k, M_g^\times(R_k) \rangle + \sum_{k=1}^{N-1} \frac{1}{2} \langle \Sigma^\times_k, F_{k-1}^T (J\Omega_{k-1})^\times F_{k-1} - (J\Omega_k)^\times \rangle.$$
Accounting for the forcing terms, the discrete Lagrange-d’Alembert principle (14) is expressed as
\[
\sum_{k=1}^{N-1} \left( \frac{1}{2} \langle \Sigma_k^x, F_{k-1}^T (J \Omega_{k-1})^x F_{k-1} - (J \Omega_k)^x + h M_g^x (R_k) \rangle + \frac{h}{2} \langle \Sigma_k^x, \tau_k^x \rangle \right) = 0.
\]
Using the identity \( F \nu^x F^T = (F \nu)^x \), for \( F \in \text{SO}(3) \) and \( \nu \in \mathbb{R}^3 \), the above expression gives
\[
J \Omega_{k+1} = F_k^T J \Omega_k + h M_g (R_{k+1}) + h \tau_k.
\]  
This is the discrete counterpart to the dynamics equations (2). This corresponds to the *forced discrete Euler-Lagrange equations* (see, e.g., ref. [32]) for a system with left-invariant kinetic energy.

C. Summary of Discrete Equations of Motion

To summarize, the continuous equations of motion (1)-(2) are approximated using the following discrete numerical integration scheme [29, 30]:

\[
F_k J - J F_k^T = h (J \Omega_k)^x,
\]
\[
R_{k+1} = R_k F_k,
\]
\[
J \Omega_{k+1} = F_k^T J \Omega_k + h \left( M_g (R_{k+1}) + \tau_k \right).
\]
Here \( J = \frac{1}{2} \text{trace}[J] I - J \) is a modified inertia matrix, \( h \) is the fixed step size (meaning that \( t_{k+1} = t_k + h \)), and
\[
\tau_k := \frac{1}{2} \left( \tau(R_k, \Omega_k, t_k) + \tau(R_{k+1}, \Omega_k, t_{k+1}) \right).
\]
Using the angular velocity \( \Omega_k \) at time \( t_k \) and (16), \( F_k \) is obtained by solving the implicit equation (16). This \( F_k \) along with the rotation matrix \( R_k \) at time \( t_k \), is then used to solve for \( R_{k+1} \) using (17). \( R_{k+1} \), along with \( R_k \) and \( \Omega_k \), are then used in equation (18) to solve for \( \Omega_{k+1} \). This gives a forward time map \((R_k, \Omega_k) \rightarrow (R_{k+1}, \Omega_{k+1})\) and the only implicit relation to be solved is equation (16). A simple Taylor series analysis shows that this is a first order method. The implicit equation (16) can be solved efficiently using the Newton-Raphson method with \( F_k = \exp(f_k^x) \) and solving for \( f_k \); this procedure is described in greater detail in our recent work [29]. These discrete equations are also equivalent to the ones derived in ref. [36] under a coordinate transformation.
D. Linearizing the Discrete Equations of Motion

Taking variations of the discrete kinematics equation (16) gives

\[ h(J\delta\Omega_k)^\times = \delta F_k J - J\delta F_k^T \]

\[ = (F_k\Sigma_{k+1}^\times - \Sigma_k^\times F_k)J - J(F_k\Sigma_{k+1}^\times - \Sigma_k^\times F_k)^T \]

\[ = (F_k\Sigma_{k+1}^\times J + J\Sigma_{k+1}^\times F_k^T) - (\Sigma_k^\times F_k J + J F_k^T \Sigma_k^\times) \]

\[ = ((F_k\Sigma_{k+1}^\times F_k J + (F_k J)^T (F_k\Sigma_{k+1}^\times)^\times) - (\Sigma_k^\times F_k J + (F_k J)^T \Sigma_k^\times) \]

\[ = ((\tr(F_k J)I - F_k J)F_k \Sigma_{k+1}^\times - (\tr(F_k J)I - F_k J) \Sigma_k^\times)^\times, \]

where the identity \( \eta^\times A + A^T \eta^\times = (\{\tr(A)I - A\}\eta)^\times \), for \( \eta \in \mathbb{R}^3 \) and \( A \in \mathbb{R}^{3\times3} \), has been used.

This result gives

\[ hJ\delta \Omega_k = \{\tr(F_k J)I - F_k J\}F_k \Sigma_{k+1}^\times - \{\tr(F_k J)I - F_k J\} \Sigma_k^\times, \]

that is

\[ \Sigma_{k+1} = hF_k^T (\tr(F_k J)I - F_k J)^{-1} J\delta \Omega_k + F_k^T \Sigma_k \]

\[ =: A_k \Sigma_k + B_k \delta \Omega_k. \quad (19) \]

This leads to

\[ \delta F_k = F_k\Sigma_{k+1}^\times - \Sigma_k^\times F_k = h (\{\tr(F_k J)I - F_k J\}^{-1} J\delta \Omega_k)^\times F_k. \]

Taking variations of the discrete dynamical equation (18) gives

\[ J\delta \Omega_{k+1} = \delta F_k^T J\Omega_k + F_k^T J\delta \Omega_k + h(\delta M^q_{k+1} + \delta \tau_k) \]

\[ = hF_k^T (J\Omega_k)^\times \{\tr(F_k J)I - F_k J\}^{-1} J\delta \Omega_k + F_k^T J\delta \Omega_k + h\delta M^q_{k+1}, \quad (20) \]

where

\[ \delta M^q_k =: M_k \Sigma_k. \quad (21) \]

Since the potential is a function of the configuration only, one obtains \( M_k = \mathcal{M}(R_k) \in \mathbb{R}^{3\times3} \). The control is not to be varied, i.e., \( \delta \tau_k = 0 \). This is due to the fact that the feedback control torque is based on the estimated states, which are known and not to be varied.
To summarize, the linearization of the discrete equations (16)-(18) is given by

\[
\begin{bmatrix}
\Sigma_{k+1} \\
\delta \Omega_{k+1}
\end{bmatrix} = A_k \begin{bmatrix}
\Sigma_k \\
\delta \Omega_k
\end{bmatrix},
\]

(22)

where

\[
A_k = \begin{bmatrix}
A_k & B_k \\
C_k & D_k
\end{bmatrix},
\]

\[
A_k = F_k^T,
\]

\[
B_k = hF_k^T \{ \text{tr}(F_k J) I - F_k J \}^{-1} J,
\]

\[
C_k = hJ^{-1} M_{k+1} F_k^T,
\]

\[
D_k = J^{-1} \left( h \left[ F_k^T ( J\Omega_k )^\times + h M_{k+1} F_k^T \right] \{ \text{tr}(F_k J) I - F_k J \}^{-1} J + F_k^T J \right).
\]

IV. State Estimation from Body Vector and Angular Velocity Measurements

For the situation where measurements are obtained at different sample rates from different sensors and sensor sample rates differ by rational factors, one can encounter both partial and full state measurements at measurement instants. Therefore filtering schemes for both full and partial state measurements are given in this section.

A. Uncertainty Ellipsoids

A nondegenerate uncertainty ellipsoid in \( \mathbb{R}^n \) is defined as

\[
\mathcal{E}_{\mathbb{R}^n}(x, P) = \left\{ y \in \mathbb{R}^n \bigg| (y - x)^T P^{-1} (y - x) \leq 1 \right\},
\]

where \( x \in \mathbb{R}^n \), and \( P \in \mathbb{R}^{n \times n} \) is a symmetric positive definite matrix. Here \( x \) is the center of the uncertainty ellipsoid, and \( P \) is the uncertainty matrix which determines the size and the shape of the uncertainty ellipsoid. The size of an uncertainty ellipsoid is measured by \( \text{tr}(P) \). It equals the sum of the squares of the semi principal axes of the ellipsoid.

A nondegenerate uncertainty ellipsoid centered at \((R_0, \Omega_0) \in \text{SO}(3) \times \mathbb{R}^3\) is induced from a
nondegenerate uncertainty ellipsoid in $\mathbb{R}^6$, and is described by

$$
E(R_0, \Omega_0, P) = \left\{ R \in \text{SO}(3), \ \Omega \in \mathbb{R}^3 \left\| \begin{bmatrix} \log(R_0^T R) \ 
\Omega - \Omega_0 \end{bmatrix} \in E_{\mathbb{R}^6}(0, P) \right\},
$$

where $P \in \mathbb{R}^{6 \times 6}$ is a symmetric positive definite matrix and $\log(\cdot)$ denotes the matrix logarithm.

Equivalently, an element $(R, \Omega) \in E(R_0, \Omega_0, P)$ can be written as

$$
R = R_0 \exp(\zeta^x),
\Omega = \Omega_0 + \delta \Omega,
$$

for some $x = [\zeta^T, \delta \Omega^T]^T \in \mathbb{R}^6$ satisfying $x^T P^{-1} x \leq 1$.

A degenerate uncertainty ellipsoid in $\mathbb{R}^n$ is a generalization of nondegenerate uncertainty ellipsoids that has at least one but less than $n$ of its principal axes infinitely long. It is given by

$$
E_{\mathbb{R}^n}(x, M) = \left\{ y \in \mathbb{R}^n \left| (y - x)^T M (y - x) \leq 1 \right. \right\},
$$

where $x \in \mathbb{R}^n$, and $M \in \mathbb{R}^{n \times n}$ is a symmetric positive semidefinite matrix. Note that if $M$ is actually definite, then $E_{\mathbb{R}^n}(x, M) = E_{\mathbb{R}^n}(x, M^{-1})$. Again, $x$ is the center of the uncertainty ellipsoid and $M$ is the degenerate uncertainty matrix, which determines the size of the finite-length principal axes and the shape of the degenerate uncertainty ellipsoid.

A degenerate uncertainty ellipsoid centered at $(R_0, \Omega_0) \in \text{SO}(3) \times \mathbb{R}^3$ is induced from a degenerate uncertainty ellipsoid in $\mathbb{R}^6$ according to

$$
E(R_0, \Omega_0, M) = \left\{ R \in \text{SO}(3), \ \Omega \in \mathbb{R}^3 \left\| \begin{bmatrix} \log(R_0^T R) \ 
\Omega - \Omega_0 \end{bmatrix} \in E_{\mathbb{R}^6}(0, M) \right\},
$$

where $M \in \mathbb{R}^{6 \times 6}$ is a symmetric positive semidefinite matrix.

**B. Uncertainty Ellipsoid Based on Measurements**

Consider the angular velocity $\Omega$ to be measured directly, as is usually the case. Denote by $\bar{\Omega}$ its measured value. The angular velocity measurement error $\delta \bar{\Omega}$ is assumed to be bounded by the ellipsoid

$$
\delta \bar{\Omega} \in E_{\mathbb{R}^3}(0, T), \text{ where } \Omega = \bar{\Omega} + \delta \bar{\Omega}.
$$

(23)
Attitude is measured indirectly, through direction measurements. Assume that there are \( m \) fixed points in the inertial reference frame, no two of which are co-linear, that are measured in the body frame. Denote the known direction of the \( i \)th point in the inertial reference frame as \( e^i \in S^2 \) where \( S^2 \) denotes the two sphere (embedded in \( \mathbb{R}^3 \)). The corresponding vector is represented in the body fixed frame as \( b^i \in S^2 \). Since only directions are measured, normalize \( e^i \) and \( b^i \) so that they have unit lengths. The \( e^i \) and \( b^i \) are related by a rotation matrix \( R \in \text{SO}(3) \) that defines the attitude of the rigid body:

\[
e^i = Rb^i,
\]

for all \( i \in \{1, 2, \ldots, m\} \). Assume that \( e^i \) is known accurately and that \( b^i \) is measured in the body fixed frame. Let \( \tilde{b}^i \in S^2 \) denote the measured direction vector, which contains measurement errors. Assume that the error in measuring \( b^i \) is given by \( \nu^i \) as

\[
b^i = \exp\left(\left(\nu^i\right)\times\right)\tilde{b}^i,
\]

where \( \nu^i \) is bounded by

\[
\nu^i \in \mathcal{E}_{\mathbb{R}^3}(0, S^1). \tag{24}
\]

Based on the measured vector \( \tilde{b}^i \) an estimate of the rotation matrix \( \tilde{R} \in \text{SO}(3) \) is obtained. The vector estimation errors are given by

\[
e^i - \tilde{R}\tilde{b}^i, \quad i = 1, \ldots, m.
\]

The weighted Euclidean norm of these errors is given by the error functional,

\[
\mathcal{J}(\tilde{R}) = \frac{1}{2} \sum_{i=1}^{m} w_i (e^i - \tilde{R}\tilde{b}^i)^T (e^i - \tilde{R}\tilde{b}^i),
\]

\[
= \frac{1}{2} \text{tr}\left[ (E - \tilde{R}\tilde{B})^T W (E - \tilde{R}\tilde{B}) \right],
\]

where \( E = [e^1, e^2, \ldots, e^m] \in \mathbb{R}^{3 \times m}, \tilde{B} = [\tilde{b}^1, \tilde{b}^2, \ldots, \tilde{b}^m] \in \mathbb{R}^{3 \times m}, \) and \( W = \text{diag}(w^1, w^2, \ldots, w^m) \in \mathbb{R}^{m \times m} \) has a weighting factor for each measurement. As per our earlier assumption, \( m \geq 2 \). If \( m = 2 \), the cross product of the two measured unit vectors \( \tilde{b}^1 \times \tilde{b}^2 = \tilde{b}^3 \) is treated as a third measured direction, with the corresponding unit vector in the inertial frame taken to be \( e^3 = e^1 \times e^2 \). The
attitude determination problem then consists of finding $\tilde{R} \in \text{SO}(3)$ such that the error functional $J$ is minimized:

$$
\tilde{R} = \arg \min_{C \in \text{SO}(3)} J(C)
$$

(25)

The problem (25) is known as Wahba’s problem [37]. It was solved shortly after it was posed [38]. A solution, known as the QUEST algorithm [39], is expressed in terms of quaternions. A solution without using generalized attitude coordinates has also been found [17]. A necessary condition for optimality of (25) is given by [17]

$$
L^T \tilde{R} = \tilde{R}^T L,
$$

(26)

where $L = EW \tilde{B}^T \in \mathbb{R}^{3 \times 3}$.

The unique minimizing solution to the attitude determination problem (25), which satisfies (26), is given by [17]

$$
\tilde{R} = SL, \quad S = Q_L \sqrt{(R_L R_L^T)^{-1}} Q_L^T,
$$

(27)

where

$$
L = Q_L R_L, \quad Q_L \in \text{SO}(3),
$$

(28)

and $R_L$ is upper triangular and invertible; this is the QR decomposition of $L$. The symmetric positive definite (principal) square root is used in (27).

Using existing results [18], the uncertainty matrix in the case of full state measurements (simultaneous body vector and angular velocity measurements) is given by

$$
\tilde{P} = \left( \sum_{i=1}^{m} \sqrt{\text{tr} \left( K_i S_i (K_i^i)^T \right) + \sqrt{\text{tr} (T)}} \right) \left( \sum_{i=1}^{m} \frac{H_1 K_i S_i (K_i^i)^T H_1^T}{\sqrt{\text{tr} \left( K_i S_i (K_i^i)^T \right)}} + \frac{H_2 T H_2^T}{\sqrt{\text{tr} (T)}} \right),
$$

(29)

where

$$
K_i = -w_i \left( \text{tr} \left( \tilde{R}^T L \right) I_{3 \times 3} - \tilde{R}^T L \right)^{-1} \left( \text{tr} \left( \tilde{b}^T (e_i)^T \tilde{R} \right) I_{3 \times 3} - \tilde{b}^T (e_i)^T \tilde{R} \right),
$$

(30)

with $H_1 = [I_{3 \times 3}, 0_{3 \times 3}]^T$ and $H_2 = [0_{3 \times 3}, I_{3 \times 3}]^T$. If body vectors are measured, but no velocity measurement is available, then the degenerate uncertainty ellipsoid this measurement defines has
the degenerate uncertainty matrix

\[
\tilde{M} = \left( \sum_{i=1}^{m} \sqrt{\text{tr} \left( K_i S_i (K_i)^T \right)} \right)^{-1} H_1 \left( \sum_{i=1}^{m} \frac{K_i S_i (K_i)^T}{\sqrt{\text{tr} \left( K_i S_i (K_i)^T \right)}} \right)^{-1} H_1^T.
\]  

(31)

If instead angular velocity is measured, but no attitude information is available, the corresponding degenerate uncertainty ellipsoid is given by the following degenerate uncertainty matrix

\[
\tilde{M} = H_2 T^{-1} H_2^T.
\]  

(32)

Hence the measured uncertainty ellipsoid is given by

\[
E(\tilde{R}, \tilde{\Omega}, \tilde{M}),
\]

and is calculated for all possible cases of full or partial state measurements, as follows.

Case 1: Full state (body vector and angular velocity) measurements: \(\tilde{M}^{-1} = \tilde{P}\) is given by (29), \(\tilde{R}\) is given by (27)-(28), and \(\tilde{\Omega}\) is measured directly.

Case 2: Attitude (body vector) measurements only: \(\tilde{M}\) is given by (31), \(\tilde{R}\) is given by (27)-(28), and \(\tilde{\Omega}\) is redundant in specifying \(E(\tilde{R}, \tilde{\Omega}, \tilde{M})\).

Case 3: Angular velocity measurements only: \(\tilde{M}\) is given by (32), \(\tilde{\Omega}\) is measured directly, and \(\tilde{R}\) is redundant in specifying \(E(\tilde{R}, \tilde{\Omega}, \tilde{M})\).

Remark 1  Note that in a practical implementation \(E = [e_1 e_2 \ldots e_m]\) and \(S^i\) may have different values for different measurements. This is due to the fact that at different attitudes, one may have different directions available for observation and the quality of a measurement of a certain direction can depend on the attitude as most direction measuring instruments have limited field of view. The angular velocity measurement error bound, given by \(T\), may also change between measurements.

C. Flow-propagated Uncertainty Ellipsoid

At some instant \(t = t_0\), let the system state be in the uncertainty ellipsoid

\[
(R, \Omega) \in \mathcal{E}(R_0, \Omega_0, P_0).
\]

As in ref. [18], the discrete equations of motion from section III are used to propagate the center of this ellipsoid; and the linearized discrete equations are used to propagate the uncertainty matrix. Using the linearized equations to propagate the uncertainty matrix ensures that the resulting
uncertainty bounds remain ellipsoidal. The validity of using the linearized equations to update the uncertainty bound depends on: (1) the size of $\mathcal{E}(R_0, \Omega_0, P_0)$, i.e. $\text{tr}(P_0)$ (the smaller the ellipsoid is, the better this approximation); and (2) the time duration of propagation, which is given by the fastest sensor sample rate. Following this approach, at $t_N = t_0 + Nh$ the state lies in the flow-propagated uncertainty ellipsoid

$$(R, \Omega) \in \mathcal{E}(R^f, \Omega^f, P^f),$$

where $R^f = R_N$ and $\Omega^f = \Omega_N$ are given by equations (16)-(18), with initial condition $(R_0, \Omega_0)$. The updated uncertainty matrix $P^f = P_N$ is given by the discrete dynamics

$$P_j = A_{j-1}A_{j-2} \cdots A_1 P_0 (A_{j-1}A_{j-2} \cdots A_1)^T,$$  \hspace{1cm} (33)

where $A_k$ is the linear flow matrix given by equations (22).

![Fig. 1 Inclusion of the intersection of two ellipsoids by a minimal trace ellipsoid, depicted in 2D. Two nondegenerate ellipsoids intersecting (left) and a degenerate ellipsoid intersecting a nondegenerate ellipsoid (right). The symbol ‘+’ is used to mark the center of an ellipsoid.](image)

**D. Intersection of Measurement and Flow Uncertainty Ellipsoids as Filter**

Let $\mathcal{E}_{\mathbb{R}^n}(x_1, P_1)$ and $\mathcal{E}_{\mathbb{R}^n}(x_2, P_2)$ be two non-degenerate ellipsoids with nonempty intersection. Consider the problem of finding an ellipsoid $\mathcal{E}_{\mathbb{R}^n}(\hat{x}, \hat{P})$ satisfying

$$\min \text{tr}(\hat{P}), \text{ such that } \mathcal{E}_{\mathbb{R}^n}(x_1, P_1) \cap \mathcal{E}_{\mathbb{R}^n}(x_2, P_2) \subset \mathcal{E}_{\mathbb{R}^n}(\hat{x}, \hat{P}).$$  \hspace{1cm} (34)

Ellipsoid $\mathcal{E}_{\mathbb{R}^n}(\hat{x}, \hat{P})$ will then be the minimum trace ellipsoid that contains the intersection of the ellipsoids $\mathcal{E}_{\mathbb{R}^n}(x_1, P_1)$ and $\mathcal{E}_{\mathbb{R}^n}(x_2, P_2)$. This is depicted in Figure 1 (left) for ellipses in the plane.
The solution to (34) is given by [40]

\[
\dot{x} = x_1 + L(q_0)(x_2 - x_1),
\]

\[
\dot{P} = \beta(q_0)((I - L(q_0))P_1(I - L(q_0))^T + q_0^{-1}L(q_0)P_2L(q_0)^T),
\]

where

\[
L(q) = P_1(P_1 + q^{-1}P_2)^{-1},
\]

\[
\beta(q) = 1 + q - (x_2 - x_1)^T P_1^{-1}L(q)(x_2 - x_1),
\]

and \( q = q_0 \) solves the equation

\[
\frac{\text{tr}(U^{-1}P_2U(\Lambda + qI)^{-2})}{\text{tr}(U^{-1}P_2U(\Lambda + qI)^{-1})} = \frac{\beta'(q)}{\beta(q)},
\]

using the eigendecomposition \( P_1^{-1}P_2 = U\Lambda U^{-1} \). A sufficient condition for (39) to have a solution is

\[
(1 - (x_2 - x_1)^TP_1^{-1}(x_2 - x_1))\text{tr}(P_1) - \text{tr}(P_1P_2^{-1}P_1) \leq 0,
\]

and if this condition is violated, \( q_0 = 0 \) is the optimal solution. Using the Newton-Raphson method with initial guess \( q_0 = 0 \), one can numerically solve (39); Figure 1 was obtained using this approach.

![Time evolution](image)

**Fig. 2** The filtering procedure depicted with planar ellipses.

Since the flow-propagated uncertainty ellipsoid is nondegenerate, but the measurement-based uncertainty ellipsoid may be degenerate when attitude and angular velocity measurements are made
at different rates, the intersection-based filtering scheme given by (34) is generalized to

$$\min \text{tr}(\hat{P}) \text{ such that } E_{\mathbb{R}^n}(x_1, P_1) \cap E_{\mathbb{R}^n}(x_2, M_2) \subset E_{\mathbb{R}^n}(\hat{x}, \hat{P}).$$  \hspace{1cm} (40)$$

This is an intersection of a non-degenerate (flow) ellipsoid with a degenerate (measurement) ellipsoid, as illustrated in Figure 1 (right). This generalizes the deterministic filtering scheme to the case of partial state measurements \cite{40}, as is encountered during multi-rate measurements when measurements for different state variables are available at different rates. Rewriting the intersection results for this general case, one obtains

$$\hat{x} = x_1 + L(q_0)(x_2 - x_1),$$  \hspace{1cm} (41)$$

$$\hat{P} = \beta(q_0)(I - L(q_0))P_1,$$  \hspace{1cm} (42)$$

where

$$L(q) = q(P_1^{-1} + qM_2)^{-1}M_2,$$  \hspace{1cm} (43)$$

$$\beta(q) = 1 + q - (x_2 - x_1)^T P_1^{-1} L(q)(x_2 - x_1),$$  \hspace{1cm} (44)$$

and \(q = q_0\) solves the equation

$$\frac{\text{tr}\left((P_1^{-1} + qM_2)^{-1}(I - qM_2(P_1^{-1} + qM_2)^{-1})M_2P_1\right)}{\text{tr}\left((I - q(P_1^{-1} + qM_2)^{-1}M_2)P_1\right)} = \frac{\beta'(q)}{\beta(q)}. \hspace{1cm} (45)$$

As in ref. \cite{18}, this approach is used as a filter to extract the uncertainty ellipsoid at time \(t_N = t_0 + Nh\) from the intersection of the flow-propagated ellipsoid (from time \(t_0\) to \(t_N = t_0 + Nh\)) and the measurement-based ellipsoid (at time \(t_N = t_0 + Nh\)). Formulated in the language of uncertainty ellipsoids on \(\text{SO}(3) \times \mathbb{R}^3\), this involves solving the minimization problem

$$\min \text{tr}(\hat{P}) \text{ such that } E(R^f, \Omega^f, P^f) \cap E(\tilde{R}, \tilde{\Omega}, \tilde{M}) \subset E(R, \Omega, \hat{P}).$$

The filtered states \((\hat{R}, \hat{\Omega})\) form the updated state estimates at time \(t_N\). This filtering procedure is illustrated in Figure 2. This problem of intersecting uncertainty ellipsoids on \(\text{SO}(3) \times \mathbb{R}^3\) needs to be converted to an equivalent problem of intersecting uncertainty ellipsoids on \(\mathbb{R}^6\) in order to use this filtering approach. Choose the center of the flow-propagated ellipsoid as the origin in \(\mathbb{R}^6\), i.e.,

$$E_{\mathbb{R}^6}(0, P^f).$$
This corresponds to locally linearizing the nonlinear state space of attitude motion, \( \text{SO}(3) \times \mathbb{R}^3 \), at \((\mathbf{R}^f, \Omega^f)\). The center of the measurement-based ellipsoid in \( \mathbb{R}^6 \) is then obtained from this local linearization, and is given by

\[
x = \begin{bmatrix}
\zeta \\
\delta \Omega \\
\end{bmatrix} = \begin{bmatrix}
\log \left( (\mathbf{R}^f)^T \tilde{\mathbf{R}} \right) \\
\tilde{\Omega} - \Omega^f
\end{bmatrix}.
\]

Thus the equivalent problem in \( \mathbb{R}^6 \) is the following minimization problem

\[
\min \text{tr}(\hat{P}) \text{ such that } \mathcal{E}_{\mathbb{R}^6}(0, P^f) \cap E_{\mathbb{R}^6}(x, \tilde{M}) \subset \mathcal{E}_{\mathbb{R}^6}(\hat{x}, \hat{P}).
\] (46)

Using (41)-(44) with \( x_1 = 0, x_2 = x, P_1 = P^f, \) and \( M_2 = \tilde{M} \), one obtains the filter-updated uncertainty ellipsoid in \( \mathbb{R}^6 \) as the solution to (46). The updated uncertainty ellipsoid in \( \text{SO}(3) \times \mathbb{R}^3 \) is given by

\[
\mathcal{E}(\hat{R}, \hat{\Omega}, \hat{P}),
\]

where the center (and updated state estimate) \((\hat{R}, \hat{\Omega})\) \( \in \text{SO}(3) \times \mathbb{R}^3 \) is obtained from \( \hat{x} = [\zeta^T, \delta \Omega^T]^T \in \mathbb{R}^6 \) as

\[
\hat{R} = R^f \exp(\hat{\zeta} \times),
\]

\[
\hat{\Omega} = \Omega^f + \delta \hat{\Omega}.
\] (47)

V. Attitude Feedback Control using State Estimates based on Multi-rate Measurements

In this section a compensator is constructed where the tracking control is based on that described in section II but instead of the true state (of which there is no exact knowledge), feedback of the estimated state as obtained from the scheme in section IV is implemented, with the filter applied at each measurement to improve the accuracy of the estimate. These measurements are assumed multi-rate, i.e., body vector measurements and angular velocity measurements may not be available at the same frequency.

The following compensator algorithm is proposed, based on feedback of state estimates obtained from the estimation scheme of section IV in the attitude state tracking control scheme of section II.

Algorithm: Let \( t_j \in \mathbb{R} \) be the initial time and \( t_{j+1} > t_j \) the time of the next measurement. Choose the time step \( 1 \gg h > 0 \) such that \( (t_{j+1} - t_j)/h \) is an integer. Let \( \hat{R}(t_j) \) and \( \hat{\Omega}(t_j) \) be the
initial estimate of the state with initial estimated uncertainty matrix $\hat{P}(t_j)$. Let the control torque (7) be based on the estimated state $(\hat{R}(t), \hat{\Omega}(t))$, i.e.

$$
\tau = -L\hat{\omega} + J\hat{Q}^T\hat{\Omega}_d + (\hat{Q}^T\Omega_d)^{x} J\hat{Q}^T\Omega_d + \Phi'(\text{tr}(K - K\hat{Q})) \sum_{j=1}^{3} k_j e_j^x \hat{Q}^T e_j - M_p(R_d\hat{Q}),
$$

where $\hat{Q} := \hat{R}^T(t)\hat{R}(t)$ and $\hat{\omega} := \hat{\Omega}(t) - \hat{Q}^T(t)\Omega_d$. The time evolution of the estimated state $(\hat{R}(t), \hat{\Omega}(t))$ is defined as follows:

1. **Propagate:** Use (16)-(18) to find the estimated state $(\hat{R}(t), \hat{\Omega}(t))$ at discrete instances in the time interval $[t_j, t_{j+1}]$; take $\hat{R}(t_j)$ and $\hat{\Omega}(t_j)$ as initial conditions. Using the linearized equations (22) along with (33) find the flow-propagated uncertainty matrix $\hat{P}(t)$ for discrete $t \in [t_j, t_{j+1}]$; take $\hat{P}(t_j)$ as initial condition. The flow uncertainty ellipsoid at time $t_{j+1}$ is $\mathcal{E}(\hat{R}(t_{j+1}), \hat{\Omega}(t_{j+1}), \hat{P}(t_{j+1}))$.

2. **Measure:** Use the measurement data to calculate the measurement-based uncertainty ellipsoid at time $t_{j+1}$ as described in section IV B.

3. **Filter:** Calculate the intersection between the flow-propagated uncertainty ellipsoid and the measurement-based uncertainty ellipsoid at time $t_{j+1}$, as described in Section IV D. The center of this ellipsoid replaces $\hat{R}(t_{j+1})$ and $\hat{\Omega}(t_{j+1})$ and its uncertainty matrix replaces $\hat{P}(t_{j+1})$.

This algorithm can then be used repeatedly using as initial time the time of the last measurement. Clearly, if the uncertainty ellipsoid stays within a bounded neighborhood of the true state, the resulting compensated system formed by combining the estimation and almost global feedback tracking schemes converges to a bounded neighborhood of the desired trajectory.

**VI. Quaternion observer-based feedback tracking scheme**

To highlight the features of our estimator-based attitude tracking scheme on SO(3), its performance is compared to that of a continuous quaternion observer-based tracking scheme. This scheme, which has the advantage of not requiring angular velocity measurements, was reported in [31], where it was claimed to provide almost global asymptotic stability in tracking. The scheme is briefly described in this section.
A quaternion in general can be represented by 
\[ q = [q_0 \ q_v]^T \in \mathbb{R}^4, \] 
where \( q_0 \in \mathbb{R} \) is the "scalar component" and \( q_v \in \mathbb{R}^3 \) is the "vector component" of the quaternion. For representing attitude, only the unit quaternions that satisfy 
\[ q^T q = q_0^2 + q_v^T q_v = 1 \] 
are important. Denote by \( q \ast p \) the quaternion product of two quaternions, given by
\[
q \ast p = [q_0 p_0 - q_v^T p_v \ (q_0 p_v + p_0 q_v + q_v \times p_v)]^T.
\] (48)

This product gives an algebraic group structure to the quaternions, of which the unit quaternions form a normal subgroup. Let \([1 \ 0 \ 0 \ 0]^T\) denote the identity element in the group of quaternions; then the inverse of a quaternion \( q = [q_0 \ q_v]^T \in \mathbb{R}^4 \) is given by
\[
q^{-1} = [q_0 - q_v^T] T / \|q\|^2,
\] (49)
such that \( q^{-1} \ast q = [1 \ 0 \ 0 \ 0]^T \). Note that the unit quaternions constitute the set \( S^3 \), the unit hypersphere embedded in \( \mathbb{R}^4 \). Every physical attitude \( R \in SO(3) \) can be represented by a pair of unit quaternions \( \pm q \in S^3 \), according to
\[
R(q) = (q_0^2 - q_v^T q_v)I + 2q_v q_v^T + 2q_0 q_v^x.
\] (50)

Note that \( R(-q) = R(q) \) as given by (50), i.e., a pair of antipodal points in the unit hypersphere \( S^3 \) are identified with a rotation matrix. Therefore the unit quaternion representation of attitude is inherently ambiguous.

Let \( q \) denote an unit quaternion vector representing the attitude \( R \) (as given by (50)), and let \( \bar{q} \) denote the unit quaternion representing the observed attitude tracking error. The actual attitude tracking error is represented by the unit quaternion \( q^e = (q^d)^{-1} \ast q \), where \( q^d \) denotes the desired unit quaternion. The unit quaternion representing the observer error is \( \tilde{q} = (\bar{q})^{-1} \ast q^e = [\bar{q}_0 \ \bar{q}_v]^T \).

The following equations describe the quaternion kinematics, the observer and the control law [31]:
\[
\dot{q} = \frac{1}{2} q \ast q^\Omega \quad \text{where} \quad q^\Omega = [0 \ \Omega^T]^T,
\] (51)
\[
\dot{\bar{q}} = \frac{1}{2} \bar{q} \ast q^\beta \quad \text{where} \quad q^\beta = [0 \ \beta^T]^T, \quad \beta = \Gamma_1 \bar{q}_v,
\] (52)
\[
\tau = -\alpha_1 q^e - \alpha_2 \bar{q}_v + JQ^T(q^e)\dot{\Omega}_d + \Omega_d^x J\Omega_d - M_g(q),
\] (53)

where, \( \alpha_1, \alpha_2 > 0 \) are controller gains and \( \Gamma_1 \) is a positive definite observer gain matrix. The proof of convergence of the continuous observer (52) in the space of unit quaternions is given in ref. [31],

23
and holds when the quaternion states are continuously measured and there are no measurement errors. The observer-generated unit quaternion, \( \bar{q} \), converges asymptotically to the unit quaternion representing the attitude tracking error, \( q^e \). Note that angular velocity measurements are not required in this continuous observer-based feedback tracking control scheme. As shown in ref. [31], the quantity \( \beta \) in (52) converges asymptotically to the angular velocity; hence \( \beta \) provides an estimate of the angular velocity for feedback. It is also claimed [31] that this observer-based tracking scheme provides stable tracking of desired attitude maneuvers; however, the proof for stability given there is applicable only to dynamics on \( S^3 \times \mathbb{R}^3 \) and not to attitude dynamics (which occurs on \( SO(3) \times \mathbb{R}^3 \)).

VII. Comparison Study of the Feedback Tracking Schemes

The purpose of this comparison study is to demonstrate that continuous (unit) quaternion feedback controllers for attitude control can waste considerable amount of time and control energy trying to overcome unwinding, even though they may make the desired attitude trajectory in the space of rigid body rotations \( SO(3) \) globally attractive. Prior work [2] has described the unwinding phenomenon, which occurs due to the two-to-one map from the unit quaternion space \( S^3 \) to the space of rotations \( SO(3) \). Unwinding results in lack of Lyapunov stability of the desired equilibrium or state trajectory, which can lead to loss of control effort and time in reaching the desired trajectory.

A. Simulation Example and Parameters

This section presents some numerical simulation results of this estimator-based control scheme applied to the practical example of a rigid satellite in circular Earth orbit at an altitude of 350 km.

For this system, the gravity-gradient torque is given by [41]

\[
M_g = 3\omega_0^2 (R_{bl}^b)^T e_3 \times JR_{bl}^b e_3,
\]

and \( \mathcal{M}_k \) is given by

\[
\mathcal{M}_k = 3\omega_0^2 \left( - (J(R_k^b)^T e_3)\times ((R_k^b)^T e_3)\times + ((R_k^b)^T e_3)\times J((R_k^b)^T e_3)\times \right),
\]

where \( \omega_0 \) is the orbital angular velocity (assumed constant) and \( R_{bl}^b \) is the rotation matrix transforming from body fixed frame to LVLH frame (local vertical local horizontal frame). In this simulation,
\[ \omega_0 = 0.0011 \, \text{s}^{-1}, \] corresponding to a circular Earth orbit at an altitude of 350 km. The discrete dynamics of \( R_{bl} \) is given by [41]

\[ R_{k+1}^{bl} = e^{-h\omega_0 e_2^*} R_k^{bl} F_k. \]

The principal moment of inertia matrix for the satellite is \( J = \text{diag}(4.2, 3.85, 4) \, \text{kg} \cdot \text{m}^2 \) which corresponds to a small satellite about 50 to 70 kg in mass. The controller parameters for this simulation are taken to be

\[ L = \text{diag}(1.6, 0.7, 1) \, \text{kg} \cdot \text{m}^2 \cdot \text{s}^{-1}, \quad K = 0.15 \, \text{diag}(1, 2, 3), \]

and \( \Phi(x) = x \). The attitude time trajectory to be tracked is given by

\[ R_d(t) = \begin{bmatrix}
  c_\theta c_\psi & -c_\theta s_\psi s_\phi - s_\theta c_\phi & -c_\theta s_\psi c_\phi + s_\theta s_\phi \\
  s_\psi & c_\phi & 0 \\
  s_\theta c_\psi & s_\theta s_\psi s_\phi - c_\theta c_\phi & s_\theta s_\psi c_\phi + s_\theta s_\phi
\end{bmatrix}, \tag{54}
\]

where \( c_\alpha = \cos(\alpha(t)) \) and \( s_\alpha = \sin(\alpha(t)) \). This attitude profile is obtained from a modified 2-3-1 Euler angle representation with

\[ \theta(t) = 0.05t - 0.2, \quad \psi(t) = 0.07t + 0.3, \quad \phi(t) = 0.03t + 0.5. \tag{55} \]

The resulting desired angular velocity time profile is

\[ \Omega_d(t) = \begin{bmatrix}
  \dot{s}_\psi + \dot{\phi} \\
  \dot{s}_\theta c_\psi - \dot{\psi} c_\phi \\
  \dot{c}_\phi c_\psi + \dot{\psi} s_\phi
\end{bmatrix}, \tag{56} \]

which gives an oscillating angular velocity profile since the angular rates are constant. The desired angular acceleration \( \dot{\Omega}_d(t) \) may be obtained easily from the \( \Omega_d(t) \) given above.

Let the initial tracking errors be

\[ Q(0) = \begin{bmatrix}
  0.7252 & 0.3297 & -0.6045 \\
  -0.6045 & 0.7252 & -0.3297 \\
  0.3297 & 0.6045 & 0.7252
\end{bmatrix}, \quad \omega(0) = \begin{bmatrix}
  -1.3650 \\
  1.6500 \\
  -1.5750
\end{bmatrix} \, \text{rad/s}. \tag{57} \]

The large initial angular velocity error (with a norm of around 152°/s) could correspond to an initially tumbling satellite at orbit insertion, after its release from a spin-stabilized launch vehicle.
Spin-stabilized launch vehicles for Earth satellites could be spinning at a rate of 1 to 2.5 rev/s; after orbit insertion, a “yo-yo mechanism” is supposed to reduce the spin rate of the satellite to about 5 to 10% of this value. One of the failure modes of analysis in such satellites deals with failure of this mechanism to fully deploy.

B. Results of Using Quaternion Observer-based Tracking Control Scheme

The desired attitude trajectory \( R_d(t) \) in (54) is converted to a corresponding continuous \( C^2 \) desired quaternion trajectory \( q^d(t) \), using the quaternion product of the component rotations of the 2-3-1 Euler angle representation used to obtain \( R_d(t) \). Note that there are other algorithms that would do this transformation [42] such that the “scalar component” of the obtained quaternion is always positive; this however would give rise to a non-\( C^2 \) trajectory in \( S^3 \) and the scheme given by (51)-(53) would not be applicable in that case. This is an inherent problem with generating a desired quaternion trajectory from a desired attitude trajectory due to the ambiguity of the quaternion representation of attitude. The initial tracking errors are as given in (57), while the desired trajectory on TSO(3) is given by (54)-(56). The initial value of the observer state \( \bar{q} \) in equation (52) is chosen randomly such that it is within a principal angle rotation of \( \pi/4 \) radians away from the initial value of the quaternion tracking error \( q^e \). A Runge-Kutta solver is used to numerically integrate the equations for the feedback system and the continuous observer, while projecting onto the space of unit quaternions \( S^3 \) at each time step. Note that the continuous quaternion observer design in [31] does not take into account measurement noise and assumes that the true quaternion state is exactly known.

The results of applying this scheme with \( \alpha_1 = 0.9, \alpha_2 = 0.84 \) and \( \Gamma_1 = 0.7I \) are shown in figures 3 to 5. In Figure 3, the unit quaternion tracking errors \( q^e = (q^e_0, q^e_1, q^e_2, q^e_3) \) and observer errors \( \hat{q} = (\hat{q}_0, \hat{q}_1, \hat{q}_2, \hat{q}_3) \) are given in components. From the time responses in these plots, observe that while the quaternion tracking error \( q^e \) and estimation error \( \hat{q} \) for the feedback system are attracted to \( (1, 0, 0, 0) \), they are repelled from the error quaternions \( q^e = \hat{q} = (-1, 0, 0, 0) \) which are unstable for the feedback dynamics. Since any antipodal quaternion pair \( (q \text{ and } -q) \) corresponds to the same physical attitude, this results in homoclinic orbits in SO(3) at \( Q(q^e) = R_d(q^d)^T R(q) = I \).
Fig. 3 Time plots of the tracking error quaternion components (left) and the estimate error quaternion components (right) for the quaternion-based tracking scheme.

Moreover, a control law that assigns different control values and/or exhibits different behavior for the two quaternions $q^e$ and $-q^e$ (corresponding to the same physical attitude tracking error), is not well-defined on $\text{TSO}(3)$.

Fig. 4 Time plots of the corresponding attitude tracking error norm (left) and the angular velocity tracking error norm (right) for the quaternion-based tracking scheme.

Figure 4 shows the time response of the corresponding attitude tracking error and angular velocity tracking error for this quaternion tracking scheme. The effects of unwinding of this quaternion-based tracking scheme on the state space of attitude motion $\text{TSO}(3)$ is clearly visible in these plots. These effects show up as transient oscillations in the attitude and angular velocity tracking errors in Figure 4, as this tracking control scheme ensures that the quaternion errors are repelled
away from \( q^e = (−1, 0, 0, 0) \in \mathbb{S}^3 \), which represents \( Q(q^e) = I \in \text{SO}(3) \). Note that the pair of unit quaternion tracking errors \( \pm q^e = \pm(1, 0, 0, 0) \) in Figure 3 corresponds to the identity matrix \( (R_t^T R)(q^e) = Q(q^e) = I \) for attitude tracking error in Figure 4.

Finally, the time profile of the norm of the control torque and the integral control effort are given in Figure 5. The control torque for the quaternion observer-based tracking scheme is seen to oscillate over a long time period after the start of the maneuver, as it follows the oscillating attitude and angular velocity tracking errors due to unwinding. This, in turn, leads to an overall control effort of around 52.1 Nms over the two minute duration of the maneuver, as shown on the time plot on the right side of Figure 5.

C. Results of Using Almost Global Tracking Control Scheme on SO(3)

With the given simulation parameters for this rigid satellite, the Lie group variational integrator (16)-(18) is implemented to obtain the time evolution of the true state \((R(t), \Omega(t))\) with the control law (7) based on estimated states \((\hat{R}(t), \hat{\Omega}(t))\) as given by the compensator algorithm. Our numerical simulation results use a time step size of \( h = 0.01 \text{ s} \) for the variational integrator in the discrete equations of motion.

Assume that body vector measurements (for attitude determination) are available every 5.5 seconds, corresponding to about 1/1000 of the period of an orbit. Angular velocity measurements are

![Graphs](image)

**Fig. 5** Time profile of the norm of the control torque (left) and its time integral (right) for the quaternion-based tracking scheme.
made ten times as frequent as the body vector measurements, i.e. every 0.55 seconds. The total duration of the simulation is 1/50th of the period of an orbit. Direction measurements are emulated numerically with the measured inertial directions given by

\[
E = \begin{bmatrix}
0.7696 & -0.1830 & -0.8988 & 0.0720 & 0.9640 \\
-0.6378 & 0.7915 & 0.2271 & 0.9920 & 0.0263 \\
0.0316 & 0.5832 & 0.3750 & 0.1035 & 0.2647 \\
\end{bmatrix}
\]

and with errors randomly chosen in the ellipsoids given by \( S_i = (5\pi/180)^2 I \) and \( T = (\pi/180 \text{s}^{-1})^2 I \); see (23) and (24). These ellipsoids correspond to measurement error bounds of 5° in attitude and 1°/s in angular velocity about each body axis, which are somewhat worse than errors expected from coarse attitude sensors and rate gyros. The weight matrix is chosen as \( W = I_{5 \times 5} \) for simplicity. The initial error in the state estimates (between estimated states and true states, i.e., \( R(0)\hat{R}(0)^T \) and \( \Omega(0) - \hat{\Omega}(0) \)) are chosen randomly from the ellipsoid with uncertainty matrix \( \text{diag}(\pi^2, \pi^2, \pi^2, (5\pi/180)^2, (5\pi/180)^2, (5\pi/180)^2) \). This initial uncertainty matrix corresponds to error bounds of \( \pi \) radians in attitude about any body axis and 5°/s in angular velocity about any body axis.

![Fig. 6 Time plots of the attitude tracking error norm (left) and the angular velocity tracking error norm (right).](image)

In Figure 6 the norm of the error in tracking the true attitude \( R(t) \) and the tracking error in the true angular velocity \( \Omega(t) \) are plotted versus time. Observe that after the initial transient phase, the tracking errors remain bounded and small. Since every measurement has the same (non-zero)
error distribution, such a result is the best possible that can be achieved. It is important to note that even with measurement errors, unwinding is avoided. The unwinding phenomenon is a result of the inherent ambiguity of the quaternion representation that is caused by the double covering map from $S^3$ to $SO(3)$, and the use of continuous quaternion-based observers and control laws that create multiple values of control inputs for the same physical attitude. Since an element of $SO(3)$ (a rotation matrix) is used to uniquely represent the attitude in the estimator and almost global tracking control law, unwinding is prevented.

The performance of the estimation approach is illustrated in Figure 7. Since the error bounds on the measurements here are defined as constants such a non converging behavior of the estimation
error is expected; stricter bounds will produce better estimation. Figure 7 (right) shows the time evolution of the size, given by the trace of the uncertainty matrix $\hat{P}$, of the flow-propagated uncertainty ellipsoid according to the compensator algorithm (and thereby implicitly (33)). In Figure 8 (left), the norm of the applied control torque as given by the compensator is plotted against time. After initial transients, the torque remains bounded since it has to mainly compensate for the measurement errors. Figure 8 (right) plots the time integral of this norm over the maneuver duration, which is a measure of the control energy expended.

D. Summary of Results From Comparison Study

The quaternion observer-based tracking control scheme given by (51)-(53) and the attitude estimator-based almost global attitude tracking control scheme given by the compensator algorithm are applied to the model of a small satellite in low-Earth orbit. The desired attitude time trajectory is given by equations (54)-(56) and the initial tracking errors are given by equation (57). Although the tracking errors are initially identical in figures 4 and 6, the errors in Figure 4 show larger oscillations in these errors. It is to be noted that the control effort in Figure 8 is initially larger than the control effort in Figure 5 due to the large initial angular velocity tracking error. However, the control effort for the quaternion observer-based tracking scheme shown in Figure 5 keeps oscillating significantly longer as it follows the oscillating attitude and angular velocity tracking errors due to unwinding. This leads to a much larger control effort (given by the time integral of the control torque norm during the maneuver) for the quaternion observer-based tracking scheme, as shown in the time plots on the right sides of figures 5 and 8.

To highlight these differences, the norm of the control torque and its time integral are plotted for both tracking schemes over the approximately two minute (120.8 s) simulation period of the maneuver, on the same time scale in Figure 9. The integral of the norm of the control torque is a measure of the overall control energy spent during the maneuver. At the end of the simulation period, the value of this integral is about 52.1 Nms for the quaternion observer-based tracking scheme, as compared to about 18 Nms for the estimator-based almost global tracking scheme. This shows that despite producing larger initial (transient) control torque, the estimator-based almost
global tracking scheme on SO(3) requires only about one-third the control energy as the quaternion observer-based tracking scheme for this attitude tracking maneuver. This happens even though the quaternion observer-based tracking scheme assumes perfect state measurements, while the estimator-based tracking on SO(3) has to compensate for bounded measurement noise that is filtered out by the estimator to produce state estimates for feedback.

In summary, the quaternion-based feedback system (51)-(53) exhibits unwinding when applied to track attitude maneuvers. This feedback system, when projected on the state space SO(3) × \mathbb{R}^3 of attitude motions, initially diverges away from the desired attitude trajectory when the quaternion tracking error gets close to \( q^e = [-1, 0, 0, 0]^T \). However, this quaternion error vector corresponds to zero attitude tracking error, i.e., \( R_T^T R = I \) or \( R = R_d \). Unwinding is unstable behavior (in the sense of Lyapunov), although the desired attitude equilibrium or state trajectory may be attractive, as in this case. As explained in ref. [25], this results from the mapping of the stable equilibrium \((q^*, \omega) = ([1, 0, 0, 0]^T, [0, 0, 0]^T)\) and the unstable equilibrium \((q^*, \omega) = ([−1, 0, 0, 0]^T, [0, 0, 0]^T)\) in the state space \( S^3 \times \mathbb{R}^3 \) to the single equilibrium \( (R_T^T R, \omega) = (I, [0, 0, 0]^T) \) in the state space SO(3) × \mathbb{R}^3 of attitude motion. This resulting equilibrium of the attitude tracking error dynamics becomes a saddle point with stable and unstable manifolds connected by homoclinic paths [25].

Though the states are eventually attracted to the desired attitude and angular velocity profile on...
SO(3) × R^3, the quaternion observer-based feedback tracking scheme wastes significantly more time and effort than the estimator-based feedback attitude tracking scheme, which is designed to achieve almost global asymptotic stability on SO(3) × R^3.

VIII. Conclusion

In this paper, an attitude and angular velocity estimation scheme for controlled rigid body motion was presented, and it was used to provide state estimates for feedback in an attitude and angular velocity feedback tracking control scheme. These control and state estimation schemes can be applied to attitude tracking control of rigid bodies in arbitrary rotational motion, since they are based on a global representation of attitude motion. Estimated states were updated from discrete-time measurements using a filter applied to update state estimates after every full or partial state measurement. Angular velocity measurements were assumed to be at a different rate than attitude (direction vector) measurements. Measured direction vectors to known spatial directions were used to determine the attitude using a method that solves the attitude determination problem directly on the group of rigid body rotations. This approach combines our earlier developments in almost global feedback attitude tracking control and attitude and angular velocity estimation using state measurements that have known ellipsoidal uncertainty bounds. For the propagation of estimated states between measurements, a Lie group variational integrator was used. Lie group variational integrators maintain the structure of the group of rigid body attitude motion and numerically conserve quantities if they are conserved by the (continuous) dynamics. A linearization of this integrator about the estimated state trajectory was used to propagate the uncertainty matrix. This estimator does not depend on the statistical distribution of noise in measured states and initial conditions. Numerical simulations confirmed that the proposed estimator-based compensator can track the desired attitude and angular velocity trajectory for an initially tumbling rigid body, based on noisy measurements with known error bounds. A comparison of this scheme with a continuous quaternion observer-based feedback tracking scheme (assuming noise-free attitude measurements) was carried out for tracking the same desired attitude maneuver. This comparison showed that the quaternion-based feedback tracking scheme exhibited unwinding behavior, whereby the desired
attitude trajectory was rendered attractive but unstable. This resulted in the quaternion feedback scheme requiring larger overall control effort and longer time to reach the desired attitude trajectory in comparison to the estimator-based almost global attitude tracking scheme on the state space of rigid body rotational motion, even though the initial (transient) control torque for the almost global attitude tracking scheme was slightly larger. Introduction of measurement noise in the quaternion feedback tracking scheme is expected to aggravate the instability due to unwinding. In conclusion, the results in this paper demonstrate the advantages of using the natural representation of physical attitude (i.e., rotation matrices) when tracking arbitrary attitude maneuvers using continuous state feedback.

References


