The evaporative drying is the main step of many methods of the ceramic material manufacturing. One of the main problems here is the cracking and possible fracture due to the shrinkage during drying. The change of mechanical properties of the body during the process presents a real challenge to study the fracture of the drying body. The methods of the mechanics of composites are used here to estimate the effective properties of ceramics during manufacturing. Two models of the drying body are used: viscoelastic liquid matrix containing a set of elastic solid inclusions, that corresponds to the first stage of the process, and liquid inclusions distributed in the elastic matrix, corresponding to the final stage, when the solid network is formed. The obtained results are in good agreement with experimental results available in the literature.

1. Motivation

The evaporative drying has been intensively investigated during the last few years mainly due to its significant role in the development of new ceramic technologies like sol–gel method. In combination with forming methods, e.g. a slip casting, dip coating, spin coating, etc., it permits to obtain new homogeneous high-purity materials and shaping routes (fibers, thin films, near net shape objects, etc.) which are studied to the requirements of the modern materials science. At the same time the evaporative drying is a classical ceramic technique which has been phenomenologically studied from the ancient ages. One of the main difficulties of this process is cracking and possible fracture due to the shrinkage while drying or baking. A comprehensive survey of the stress formation during drying may be found in [1, 2]. Moreover, continuous densification of the material due to evaporation of the liquid causes a change in the effective mechanical properties. An analogy with inhomogeneous heating of the material, the properties of which are highly temperature dependent, is found. This peculiarity causes trouble to the modeling of stress formation and crack propagation in bodies during drying. Recently, an investigation of overall mechanical properties in application to the sol–gel processing was done by Jagota and Scherer [3, 4] and Wöigner et
al. [5]. A short assessment of research in this field is given in [6]. In [3, 4] the effective mechanical behaviour of a sintering composite of soft and hard particles has been modeled. However, that approach cannot be applied to the study of cracking during drying. On the other hand, experimental investigation of the variation of mechanical properties during all the process is an arduous task. Thus, the derivation of a realistic theory which could describe the macroscopic mechanical behaviour of drying ceramics is of considerable theoretical and practical interest.

2. The Model Description

It is obvious that the mechanical properties of compounds are defined mainly by their microstructure. The drying body is a good illustration to the above. We can observe the microstructure formation and evolution of mechanical properties during such a process simultaneously. The relation between the volume concentration of the solid particles and liquid in unit volume varies during the process of drying. In Fig. 1 the real structure of the drying body and the proposed computational model are presented. The situation is complicated by the fact that there is some kind of phase transition during the process. The matter is that in the first stage the material consists of unconnected solid particles distributed in the liquid and exhibits effective liquid-like behaviour. However, the concentration of solid particles increases due to the evaporation. After the percolation limit is reached the average distance between the particles becomes comparatively small and the van der Waals forces combine them in a network-like structure. From this moment the body exhibits solid-like effective properties. As far as evaporation continues from the external surface surface tension in the liquid reaches a critical value at which the liquid meniscus curvature becomes sufficient for meniscus penetration into the body. The dry shell is formed at the external surface at this moment. The stages of the process are displayed schematically in Fig. 2.

According to this model of the process we propose the following plan for the calculation of effective properties of drying bodies. Both solid and liquid materials are assumed

Fig. 1. Modelling of the microstructure formation in the drying derived material. a) SEM micrograph of sol–gel-derived KTIOPO₄ thin film (after [13]). b) Model for calculation of the average properties
to be viscoelastic having different properties. The infinite viscosity and infinite shear
modulus give the limiting cases of pure elastic solid and pure viscous liquid. The follow-
ning assumptions are used in the subsequent analysis: the relaxation time is small com-
paring to the time of concentration variation, the dimensions of the body are always
larger comparing to the mean particle diameter, and the concentration of the solid phase
during the process is a known function of time and space coordinates (complete solution
of this problem is discussed in [2]). Macroscopic mechanical behaviour is defined as an
average strain response of such a specimen to the boundary loading which would lead to
uniform or polynomial states of stress in a homogeneous medium. Here we consider the
problem of analytical prediction of such response using the effective field method [7 to 9].
This problem is sufficiently nonlinear because of the variable concentration of phases in
the ceramics during the drying. However, the proposed approach is valid in this case as
is shown in [10, 11]. A similar model in the case of linear viscoelastic matrix composites
was firstly developed by Hashin [12]. We describe here a wetted powder compact as a
two-phase composite consisting of solid particles inside the viscoelastic gel in the first
stage and liquid inclusions inside the solid network in the second stage (after the net-
work formation).
3. Effective Field Method

Firstly, let us consider an infinite linear-viscoelastic medium which satisfies the following constitutive law:

\[
\sigma_{ij}(t) = \int_0^t C^{m}_{ijkl}(t - \tau) \frac{d\varepsilon_{kl}}{d\tau} d\tau,
\]

where \(\sigma\) and \(\varepsilon\) are the stress and strain tensors, respectively and \(C^{m}\) is the fourth-rank tensor depending on time (function of relaxation). Let then a close compact region \(V\) of the medium be occupied by material which possesses different properties described by the kernel \(C^{i}_{ijkl}\). Using the Laplace transformation which is defined for an arbitrary function \(f(t)\) as

\[
\tilde{f}(s) = \int_0^\infty f(t) e^{-st} \, dt
\]
equation (3.1) can be rewritten in the following form:

\[
\tilde{\sigma}_{ij}(s) = \tilde{C}^{m}_{ijkl}(s) \tilde{\varepsilon}_{kl}(s).
\]

Hereafter, for simplicity, symbols which denote the Laplace modes of the functions will be used without overline. Let us start with the equilibrium equation in displacement for the medium with a single inclusion in the Laplace space,

\[
\nabla(C(x, s) : \nabla u(x, s)) = 0,
\]

where colon denotes for the contraction of the tensors by two indices. Rewriting the tensor \(C(x, s)\) in the form

\[
C(x, s) = C^{m}(s) + \delta C(s) V(x); \quad \delta C(s) = C^{i}(s) - C^{m}(s),
\]

where \(V(x)\) is the characteristic function of the domain \(V\), the determination of the field \(u(x, s)\) can be reduced to the corresponding integral equation,

\[
u(x, s) = u^{0}(x, s) + \int_{V} \nabla G(x - x', s) : \delta C(s) : \varepsilon(x', s) \, dx',
\]

where \(G\) is the Laplace transformation of the Green’s tensor for viscoelastic media (see Appendix 1), and \(u^{0}\) is the displacement field in a homogeneous medium without any inclusions. For \(x \in V\) equation (3.6) defines the strain field \(\varepsilon(x, s) = (\nabla u(x, s) + u(x, s) \nabla) / 2\) inside the inclusion. Using this tensor the mechanical fields outside \(V\) can be determined relatively easy. If \(V\) has an ellipsoidal shape and \(\varepsilon^{0}\) does not depend on the coordinates, then the field \(\varepsilon\) is uniform inside \(V\) as well [14, 15]. Therefore, for the deformation field in the inclusion we can get

\[
\varepsilon(x, s) = (I + P(s) : \delta C(s))^{-1} : \varepsilon^{0}(x, s) \equiv A(s) : \varepsilon^{0}(x, s),
\]

where \(I = (I_{ijkl}) = \frac{1}{2} (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\) and \(P\) is the integral over \(V\) for the second derivative of the Green’s tensor,

\[
P_{ijkl}(x, s) = \int_{V} \frac{\partial^2 G_{ik}(x, s)}{\partial x_j \partial x_l} \bigg|_{(ij)(kl)} \, dV.
\]
Next, let us consider an infinite viscoelastic body containing a spatially homogeneous random set of ellipsoidal inclusions, which occupy a system of compact regions \( \{ V_k \} \) with characteristic functions \( V_k(x) \). The equation for the Laplace mode of the strain field can be written as

\[
\varepsilon(x, s) = \varepsilon^0(x, s) + \sum_k \int_{V_k} P(x - x', s) : \delta C(x', s) : \varepsilon(x', s) V_k(x') \, dx'.
\] (3.9)

To solve the homogenization problem and develop the macroscopic equation of equilibrium with the help of (3.9), let us use a variant of the self consistent scheme, according to which

- every inclusion is considered as isolated, ‘sealed’ into the homogeneous medium with the properties of the matrix;
- the presence of surrounding inclusions is taken into account by the introduction of an effective strain field acting on the examined one.

The merits of this method and its distinctions from the traditional approaches are discussed in [9]. Let us choose one typical case realizing a random set of inclusions and consider one occupying a region \( V_k \). Let then \( \varepsilon^*(x, s) \) be a local external strain field acting on this domain. This field is defined in \( V_k \) and consists of the external field \( \varepsilon^0(x, s) \) and its disturbances due to the other inclusions.

Let now the field \( \varepsilon^*(x, s) \) coincide with \( \varepsilon^*_{(k)}(x, s) \) for \( x \in V_k \). Denoting

\[
\bar{V}(x, x') = \sum_{i \neq k} V_i(x'); \quad x \in V_k
\] (3.10)

for arbitrary \( x \in V = \bigcup_k V_k \) we get

\[
\varepsilon^*(x, s) = \varepsilon^0(x, s) + \int P(x - x', s) : \delta C(x', s) : \varepsilon(x', s) \bar{V}(x, x') \, dx',
\] (3.11)

where integration is done over the whole space. We assume that the field \( \varepsilon^*(x, s) \) has the same structure in any region occupied by inclusions. These are the main hypotheses of the effective field method. If this field is homogeneous inside each region \( V_k \) (obviously, it might vary from one inclusion to another), the relation between the fields \( \varepsilon(x, s) \) and \( \varepsilon^*(x, s) \) is defined by the relation (3.7)

\[
\varepsilon(x, s) = A(x, s) : \varepsilon^*(x, s).
\] (3.12)

The tensor function \( A(x, s) \) is defined in \( V \) and coincides with the constant tensor \( A \) from (3.7) (various for each inclusion, depending on the relations between ellipsoid semi-axes and their orientations). Substituting (3.12) into the right-hand sides of equations (3.9) and (3.11) allows to express the strain field \( \varepsilon(x, s) \) at an arbitrary point via the local external field,

\[
\varepsilon(x, s) = \varepsilon^0(x, s) + \int P(x - x', s) : C^A(x', s) : \varepsilon^*(x', s) V(x') \, dx',
\]

\[
C^A(x', s) = \delta C(x', s) : A(x, s),
\] (3.13)

and also to obtain a self-consistent integral equation for the field \( \varepsilon^*(x, s) \),

\[
\varepsilon^*(x, s) = \varepsilon^0(x, s) + \int P(x - x', s) : C^A(x', s) : \varepsilon^*(x', s) \bar{V}(x, x') \, dx'.
\] (3.14)

If the set of inclusions is a random one, the functions \( \varepsilon(x, s) \) and \( \varepsilon^*(x, s) \) are also random. Equations (3.13) and (3.14) are principal to determine the mean values of these
fields. Assuming that the local external field inside each inclusion does not depend on its
topological and material characteristics and passing to the mathematical expectations in
these equations we derive [9]

\[ \langle \varepsilon(x, s) \rangle = \varepsilon^0(x, s) + \int P(x - x') : C^A(s) : \hat{\varepsilon}^*(x', s) \, dx', \] (3.15)

\[ \hat{\varepsilon}^*(x, s) = \varepsilon^0(x, s) + \int \psi(x - x') P(x - x') : \tilde{C}^A(s) : \hat{\varepsilon}^*(x', s) \, dx'. \] (3.16)

Here \( \hat{\varepsilon}^*(x, s) = \langle \varepsilon^*(x, s) | x \rangle \) means the ensemble average of \( \varepsilon^*(x, s) \) under conditions
that \( x \in V, \tilde{C}^A(s) = n_0 \langle \nu \tilde{C}^A(x, s) \rangle, n_0 \) is the mean number of inclusions in the unit
volume, \( \nu \) is the inclusion volume, and angular brackets in the latter expression mean
the averaging over orientations and sizes of the inclusions. The function

\[ \psi(x - x') = \langle V(x) \rangle^{-1} \langle V(x, x') | x \rangle \] (3.17)

characterizes the density of the distribution of inhomogeneities surrounding a typical
inclusion. It defines the shape of the ‘correlation hole’ in which this inclusion is located.

Excluding the external field \( \varepsilon^0(x, s) \) from (3.15) and (3.16) we get the equation which
couples the effective field \( \hat{\varepsilon}^*(x, s) \) and the average field \( \langle \varepsilon(x, s) \rangle \) in the body,

\[ \hat{\varepsilon}^*(x, s) = \langle \varepsilon(x, s) \rangle + \int \varphi(x - x') P(x - x') : \tilde{C}^A(s) : \hat{\varepsilon}^*(x', s) \, dx', \] (3.18)

where \( \varphi(x - x') = 1 - \psi(x - x') \). In the following we assume that the function \( \varphi(x - x') \)
has spherical symmetry which corresponds to isotropic distribution of the inclusion centers. This function is smooth and decreasing fast to zero outside the correlation hole. If
we neglect the change of the field \( \hat{\varepsilon}^*(x, s) \) in this region, the equation (3.18) transforms
into an algebraic one,

\[ \hat{\varepsilon}^*(x, s) = \langle \varepsilon(x, s) \rangle - \Pi(s) : C^A(s) : \hat{\varepsilon}^*(x, s), \] (3.19)

where

\[ \Pi(s) = -\int P(x, s) \varphi(x) \, dx. \] (3.20)

From (3.19) for \( \hat{\varepsilon}^*(x, s) \) we find

\[ \hat{\varepsilon}^*(x, s) = D(s) : \langle \varepsilon(x, s) \rangle, \]

\[ D(s) = (I + \Pi(s) : C^A(s))^{-1}. \] (3.21)

Substituting this equation into the right-hand side of (3.15) gives

\[ \langle \varepsilon(x, s) \rangle = \varepsilon^0(x, s) + \int P(x - x', s) : C^A(s) : D(s) : \varepsilon(x', s) \, dx'. \] (3.22)

Let us contract both sides of this equation with the tensor \( C^m \) by two indices and
apply the operator of divergence. Taking into account the equilibrium equation
\( \nabla C^m(s) : \varepsilon^0(x, s) = 0 \), we find that the average mechanical fields in the whole body sa-
tify the equations

\[ \nabla C^*(s) : \langle \varepsilon(x, s) \rangle = 0, \] (3.23)

where \( C^*(s) = C^m(s) + C^A(s) : D(s) \) is the Laplace transform of the tensor of effective
viscoelastic characteristics.
4. Results and Discussion

Let us consider the drying body as a two-phases viscoelastic medium, while both the phases are isotropic and one of them is a set of spheres (inclusions) distributed in the other (matrix). In this case the Laplace mode of the relaxation kernel can be written as follows:

\[
C_{ijkl}^p(s) = K^p E_{ijkl}^1 + 2\mu^p(s) E_{ijkl}^2,
\]

where \( p = i \) for the inclusions and \( p = m \) for the matrix, \( \mu \) and \( K \) are Laplace images of the shear and bulk moduli of material, \( E^1 \) and \( E^2 \) are elements of the fourth-rank tensor basis \([16]\),

\[
E_{ijkl}^1 = \delta_{ij}\delta_{kl}, \quad E_{ijkl}^2 = I_{ijkl} - \frac{1}{3} E^1_{ijkl}.
\]

Since all inclusions have spherical shape of the same radius, tensor \( P_{ijkl} \) in (3.7) has the following form:

\[
P_{ijkl}(s) = \frac{1}{9K_e(s)} E_{ijkl}^1 + \frac{1}{2\mu_e(s)} E_{ijkl}^2,
\]

where

\[
K_e(s) = K_m + \frac{4}{3} \mu_m(s), \quad \mu_e(s) = \mu_m(s) \frac{5}{6}(3K_m + 4\mu_m(s))
\]

If we assume that the inclusions have isotropic distribution, then the correlation hole has a spherical shape and tensor \( P_{ijkl} \) in (3.20) coincides with \( P \) defined in (3.8). The macroscopic behaviour of the body, therefore, is isotropic and may be described by the two functions

\[
K^*(s) = K_m + c_i \left( \frac{1}{\delta K} + \frac{1 - c_i}{K_e(s)} \right)^{-1},
\]

\[
\mu^*(s) = \mu_m(s) + c_i \left( \frac{1}{\delta \mu(s)} + \frac{1 - c_i}{\mu_e(s)} \right)^{-1},
\]

where \( \delta \mu \) and \( \delta K \) denote the differences between the corresponding moduli of matrix and inclusions, \( c_i \) is the volume fraction of inclusions. Now the inverse Laplace transformation should be made in (4.4). In order to obtain corresponding relations in the general case the numerical methods are more preferable since the final expressions are quite cumbersome. However, the analytical relations for some simple cases are available and given in Appendix 2.

To compare our approach with the real behaviour of drying bodies, we use experimental data from [6, 17] obtained for the acid catalyzed gel made by hydrolyzing distilled tetraethylorthosilicate (TEOS) with shear modulus \( \mu_s = 1.6 \) GPa and Poisson’s ratio \( \nu = 0.2 \). The liquid is \( \text{H}_2\text{O}/\text{C}_2\text{H}_5\text{OH}/\text{HCl} = 16/4/0.01 \), that is a purely viscous mixture with viscosity \( \eta_l = 0.82 \times 10^{-3} \) Pa s and bulk modulus \( K_l = 0.226 \) GPa. It means that the solid phase of the considered body is purely elastic and the liquid is viscous (without elastic part). The effective properties of the body strongly depend on the distance from the surface of evaporation. However, the transverse shear modulus can be considered as a function of solid phase concentration only. The corresponding data are presented in Fig. 3. The initial concentration of the solid phase is 0.06. The lower curve corresponds
to a model 1 for which the solid particles are distributed in the liquid. It is in good agreement with experimental data unless 60% of the liquid was evaporated (the concentration is assumed to get changed only due to the liquid evaporation). The volume change $V/V_0$ ($V$ and $V_0$ are the current and initial volumes of the body, respectively) at this moment is equal to 0.4. The upper curve corresponds to a model 2 of the elastic

![Graph showing the dependence of the transverse shear modulus on the volume change of the body.](image1)

**Fig. 3.** Dependence of the transverse shear modulus on the volume change of the body. (Model 1 – liquid matrix, Model 2 – solid matrix, Mixed model – interpolation between Model 1 and Model 2 near the percolation point)

![Graph showing the radius dependence of material parameters in a drying cylinder.](image2)

**Fig. 4.** Example of radius dependence on the effective material parameters in a drying cylinder; (1) shear modulus, (2) viscosity
solid matrix with distributed liquid inclusion. This model gives satisfactory results starting from $V/V_0 = 0.4$. It is significant that the point of the model change corresponds to the solid phase concentration of 0.14. This point is in good agreement with the percolation limit discussed above. For the considered material the solid network is formed when the solid phase concentration is close to 0.17 [18]. It should be noted that the simplest model of Maxwell’s materials (A1.4) was used in this example.

The example considered above allows to make theoretical evaluations of the variation of the mechanical properties of the drying material. In case of a cylindrical shape of the drying body the radial dependence of the concentration profiles using the solution of the filtration problem can be applied. Examples of such curves are shown in Figs. 4 and 5. In Fig. 5 the radial dependence of solid phase concentration at an arbitrary moment of time (after the percolation limit is reached) is shown. This solution is discussed in [2] and used here to obtain the radial dependence of the effective instantaneous shear modulus $\mu$, effective viscosity $\eta$, and characteristic time $\tau_1^*$ (see equation (A2.3)). The dependence is given in relative radial coordinates $r/R_c$, where $R_c$ is the radius of the cylinder. The theoretical results shown in Figs. 4 and 5 are obtained with equations (A2.3) and (A2.4). The mechanical properties of the components were the same as in the example considered above.

5. Conclusions

In the present paper the drying material is designed as two-component composite with variable volume concentration of the phases. The proposed approach allows to describe the drying materials when a large dissimilarity in mechanical properties of the solid particles and the humectant media takes place. The model is quite simple and at the same time reasonably accurate for the use in technological applications since the results obtained are in good agreement with experimental data available in the literature. However, further study and comparison of the results with the experimental data are essen-
tial in order to investigate a more widespread set of materials. To describe it, more complicated topological and rheological models are needed.

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**Appendix 1**

The Green’s tensor for a viscoelastic medium \( G(x - x', t) \) is, by definition, the history of displacements of the point \( x' \) in response to the action of unit load at the point \( x \). For a homogeneous infinite medium with linear properties, this tensor can be represented in the form of a product of the function \( |x - x'|^{-1} \equiv 1/r \) and the tensor field \( G_0 \) defined and differentiable on the surface of the unit sphere. Its derivatives \( \nabla G, \nabla \nabla G, \) etc. can be presented in the form of the product of a singular scalar function and smooth tensor field on the unit sphere. Actually, in the spherical coordinates

\[
\nabla G = \nabla \left( \frac{1}{r} G_0 \right) = \frac{1}{r^2} \left( \nabla_0 G_0 - G_0 a_r \right),
\]

\[
\nabla \nabla G = \frac{1}{r^3} \left( \nabla_0 (\nabla_0 G_0 - G_0 a_r) - 2(\nabla_0 G_0 - G_0 a_r) a_r \right), \quad \text{etc.}
\]

Taking also into account that \( \partial a_r/\partial r = \partial a_\theta/\partial r = \partial a_\varphi/\partial r = 0 \) (orthonormal bench-mark does not vary while moving along \( r \)), it is clear that the fields in parentheses at the right-hand side of (A1.2) do not depend on \( r \). That is they are defined on a unit sphere surface as well as the tensor \( G_0 \).

For isotropic linear viscoelastic media the Laplace transformation of the Green’s tensor can be always written in the form

\[
G(r, s) = \frac{1}{r} (\alpha(s) I + \beta(s) a_r a_r).
\]

Let us consider, for example, a Maxwell material

\[
\varepsilon_{ij}^d = \frac{1}{2\mu} \varepsilon_{ij} + \frac{1}{2\eta} \sigma_{ij}^d, \quad \varepsilon_{kk} = \frac{1}{3K} \sigma_{kk}, 
\]

where the symbol \( d \) denotes the deviatoric part of the tensor, \( \mu \) and \( K \) are shear and bulk moduli of the material, respectively, and \( \eta \) is the shear viscosity. In this case

\[
\alpha(s) = \frac{\eta s + \mu}{8\pi\mu\eta s} (2 - \varkappa(s)),
\]

\[
\beta(t) = \frac{\eta s + \mu}{8\pi\mu\eta s} \varkappa(s),
\]

where

\[
\varkappa(s) = \frac{3K(\eta s + \mu) + \mu s}{3K(\eta s + \mu) + 4\mu s}.
\]
Appendix 2

The inverse Laplace transformation of the image of function \( f(s) \) of the complex variable \( s = \xi + i\eta \) is a function for which the Laplace transformation is defined by (3.2). For simplicity, we describe here both phases as Maxwell materials (see (A1) to (A4)). It should be noted, that the effective material does not satisfy this law.

Even for such a simple case more or less simple equations for the inverse Laplace transformation can only be obtained for the bulk modulus. The volume change \( \varepsilon \equiv \varepsilon_{kk} \) is connected with hydrostatic pressure \( \sigma \equiv \frac{1}{3} \sigma_{kk} \) by the following relation:

\[
\sigma(t) = K_e^* \varepsilon(t) - \kappa \int_0^t \exp \left( -\frac{t - \tau}{\tau^*} \right) \varepsilon(\tau) \, d\tau,
\]

where

\[
K_e^* = K_m + c \frac{K_e}{(1 - c) \delta K + K_e}, \quad K_e = K_m + \frac{4}{3} \mu_m, \quad \delta K = K_i - K_m,
\]

\[
\kappa = \frac{4}{3} \mu_m \frac{c(1 - c) \delta K^2 \mu_m}{(1 - c) \delta K + K_m \eta_m^*}, \quad \tau^* = \frac{(1 - c) \delta K + K_e \eta_m}{(1 - c) \delta K + K_m \eta_m}.
\]

Corresponding relations for \( \mu(s) \) are sufficiently more complicated. If one of the phases is purely elastic and the other is purely viscous, it looks as follows:

\[
\sigma_{ij}^d = 2\mu_v \varepsilon_{ij}^d + 2\eta_v \dot{\varepsilon}_{ij}^d + \kappa \int_0^t \left\{ A_1 \exp \left( -\frac{t - \tau}{\tau_1^*} \right) - A_2 \exp \left( -\frac{t - \tau}{\tau_2^*} \right) \right\} \dot{\varepsilon}_{ij}^d(\tau) \, d\tau.
\]

In the case, when the matrix is a purely elastic body and inclusions are purely viscous, i.e. \( \eta_m = \mu_i = \infty \),

\[
\mu_v^* = \left( \mu_m + (1 - c) \frac{\mu_v}{c} \right), \quad \mu_e = \mu_m \frac{5(3K_m + 4\mu_m)}{6(K_m + 2\mu_m)}, \quad \eta_v^* = 0,
\]

\[
A_1 = 1, \quad A_2 = 0, \quad \kappa = \frac{(1 - c) \mu_e^2}{c^2 \eta_i}, \quad \tau_1^* = \frac{\mu_i}{\mu_v - c\mu_m}.
\]

In the opposite case, when the elastic solid inclusions are distributed in the viscous matrix, the same relations are derived as follows:

\[
\eta_v^* = \eta_m \left\{ 1 - \frac{5c}{2 + 3c} \right\}, \quad \mu_v^* = 0,
\]

\[
A_i = \frac{3K_m(c - 1) - 20\mu_i}{\tau_1^*} - \frac{15c\mu_i K_m}{\eta_m^*}; \quad i = 1, 2,
\]

\[
\kappa = \frac{5c}{4\eta_m(2 + 3c)^2} \frac{\tau_1^* \tau_2^*}{\tau_2 - \tau_1^*} \left( \frac{1}{\tau_{1,2}} \right) = -B_1 \pm \sqrt{B_1^2 - B_2^2},
\]

\[
B_1 = \frac{15K_m + 6(1 - c)(2\mu_i - K_m)}{8\eta_m(2 + 3c)}, \quad B_2 = \frac{3(1 - c) \mu_m K_i}{(2 + 3c)2\eta_m^2}.
\]
References