Explicit cross-property correlations for anisotropic two-phase composite materials

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Received 18 December 2000; received in revised form 19 March 2001; accepted 27 March 2001

Abstract

Explicit correlations between two groups of anisotropic effective properties—conductivity and elasticity—are established for two-phase composite materials with anisotropic microstructures (non-randomly oriented inclusions of non-spherical shapes). The correlations are derived in the framework of the non-interaction approximation. The elasticity tensor is expressed in terms of the conductivity tensor in closed form. Applications to realistic microstructures, containing mixtures of diverse inclusion shapes are given. Compliance/stiffness contribution tensors of an inclusion, that characterize the inclusion’s contribution to the overall elastic response, are derived in the course of analysis; these results are of interest on their own. © 2002 Elsevier Science Ltd. All rights reserved.

Keywords: Composite materials; Elasticity; Conductivity; Effective properties; Cross-property

1. Introduction

Cross-property correlations for heterogeneous materials belong to the realm of fundamental problems of engineering science. Whereas correlations between properties governed by mathematically similar laws are straightforward (say, electric and thermal conductivities), the correlations between the elastic properties and conductivities constitute a much more complex problem. Moreover, their very existence is not obvious; besides being governed by different differential equations, they are characterized by tensors of different ranks. Their practical usefulness lies in the fact that one physical property (say, electric conductivity) may be easier to measure than the other (say, full

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set of anisotropic elastic constants). Cross-property correlations are particularly useful for anisotropic composites with high contrast between properties of constituents (fiber reinforced composites, materials with non-randomly oriented cracks/pores, etc.). In this case, the correlations involve a number of anisotropic effective constants. The importance of cross-property correlations for composites has been pointed out by Berryman and Milton (1988) and was particularly emphasized by Gibiansky and Torquato (1995).

Cross-property correlations between various effective properties of heterogeneous materials have been examined in several works. Levin (1967) interrelated the effective bulk modulus and the effective thermal expansion coefficient of the two-phase isotropic composites. Milton (1981) established cross-property bounds for the transport and the optical constants of isotropic composites. Similar bounds for the electrical and the magnetic properties were given by Cherkaev and Gibiansky (1992). The general approach to establishing various cross-property correlations was outlined by Milton (1997); see also the recent review of Markov (1999).

The conductivity–elasticity correlations were first examined by Bristow (1960) who derived an explicit connection between the effective conductivity and effective elastic moduli of a solid with cracks. The derivation was done in the framework of the non-interaction approximation and for the case of random crack orientations (overall isotropy). These correlations were further investigated in the work of Berryman and Milton (1988) on the two-phase composites, where cross-property bounds (that are narrower than the classical Hashin–Shtrikman’s ones) were established.

Results on the cross-property bounds were substantially advanced by Gibiansky and Torquato (1995, 1996a, b), who narrowed them under additional restrictions on the composite microgeometry and on the properties of constituents. Gibiansky and Torquato (1996a) also considered transversely isotropic material (fiber reinforced composite) and established bounds for two of the effective elastic constants in terms of the effective conductivities.

In contrast with the mentioned works, the present analysis is done in the framework of non-interaction approximation. It continues the analysis of Kachanov et al. (2001) on materials with cracks/pores. The cross-property correlations are established in closed form that explicitly reflects inclusion shapes and properties of the constituents.

We note, in this connection, that the non-interaction approximation, besides being rigorous at small inclusion concentration, is of fundamental importance. Indeed, it serves as a basic building block for various commonly used approximate schemes (self-consistent, differential, Mori-Tanaka’s, etc.) that place non-interacting inclusions into some sort of “effective environment” (effective field or effective matrix). Therefore, the results derived in the present work can be reformulated, in a straightforward way, in the framework of any of the mentioned schemes.

Some of the intermediate results derived in the present work are of interest on their own. They concern the compliance/stiffness contribution tensors of inclusions (that characterize the contribution of an inclusion to the overall elastic properties) and approximate representations of these tensors in terms of symmetric second rank tensors.
2. Stiffness and compliance contribution tensors of an inclusion

We consider a certain reference volume $V$ of an infinite three-dimensional medium with an inclusion of volume $V^*$—a region possessing elastic/conductive properties different from those of the surrounding material. The properties of the inclusion and of the matrix will be denoted by an asterisk and by “0”, respectively.

The compliance contribution tensor $\mathbf{H}$ of an inclusion is defined by the following relation for the overall strain per volume $V$:

$$\mathbf{bS}_{ij} = \mathbf{S}_{ijkl}^0 \mathbf{bESC}_{kl} + \mathbf{H}_{ijkl} \mathbf{bESC}_{kl};$$

(2.1)

where the second term represents the strain change $\Delta \mathbf{bS}_{ij}$ due to the presence of the inclusion. The $\mathbf{H}$-tensor depends on the inclusion shape and its elastic properties. ($\mathbf{S}^0$ is the matrix compliance tensor and $\mathbf{bESC}$ is the “remotely applied” stress, assumed to be homogeneous in the absence of the inclusion). In the case of pores, $\mathbf{H}$-tensors were derived by Kachanov et al. (1994). Similarly, the stiffness contribution tensor $\mathbf{N}$ of an inclusion is defined by the relation

$$\mathbf{bESC}_{ij} = \mathbf{C}_{ijkl}^0 \mathbf{bSI}_{kl} + \mathbf{N}_{ijkl} \mathbf{bSI}_{kl};$$

(2.2)

where $\mathbf{C}^0$ is the matrix stiffness tensor.

To derive expressions for $\mathbf{H}$- and $\mathbf{N}$-tensors in explicit form, we utilize the solution of Eshelby’s problem in the form given by Kunin and Sosnina (1971) (see, also, books of Kunin, 1983; Kanaun and Levin, 1993) that does not use Eshelby’s concept of “equivalent eigenstrain” but directly expresses a uniform field inside the ellipsoidal inhomogeneity in terms of the far field. If a uniform field (either $\mathbf{bSI}^0_{ij}$ or $\mathbf{bESC}^0_{ij}$) at infinity is prescribed, then the resulting uniform strains and stresses inside the ellipsoidal inclusion can be represented as follows:

$$\mathbf{bSI}_{ij}^{(\text{int})} = \mathbf{\Theta}_{ijkl} \mathbf{bSI}^0_{kl} \text{ if strains } \mathbf{bSI}^0_{ij} \text{ are prescribed at infinity},$$

$$\mathbf{bESC}_{ij}^{(\text{int})} = \mathbf{\Gamma}_{ijkl} \mathbf{bESC}^0_{kl} \text{ if stresses } \mathbf{bESC}^0_{ij} \text{ are prescribed at infinity};$$

(2.3)

where $\mathbf{\Theta}_{ijkl} = [\mathbf{J}_{ijkl} + \mathbf{P}_{ijmn}(\mathbf{C}_{mnkl}^* - \mathbf{C}^0_{mnkl})]^{-1}$, $\mathbf{\Gamma}_{ijkl} = [\mathbf{J}_{ijkl} + \mathbf{Q}_{ijmn}(\mathbf{S}_{mnkl}^* - \mathbf{S}^0_{mnkl})]^{-1}$ and $\mathbf{J}_{ijkl} = (\delta_{ik}\delta_{lj} + \delta_{il}\delta_{kj})/2$ is the fourth rank unit tensor. The inverse of symmetric fourth rank tensor $X_{ijkl}^{-1}$ is defined by the relation $X_{ijkl}^{-1}X_{mnkl} = (X_{ijmn}X_{ijkl}^{-1}) = \mathbf{J}_{ijkl}$.

Tensors $\mathbf{P}$ and $\mathbf{Q}$ can be expressed in terms of Eshelby’s tensor $\mathbf{s}$:

$$\mathbf{P}_{ijkl} = s_{ijmn} \mathbf{S}^0_{mnkl}, \quad \mathbf{Q}_{ijkl} = \mathbf{C}^0_{ijmn}(\mathbf{J}_{mnkl} - \mathbf{s}_{mnkl});$$

(2.4)

and tensors $\mathbf{P}$ and $\mathbf{Q}$, $\mathbf{\Theta}$ and $\mathbf{\Gamma}$ are interrelated as follows:

$$\mathbf{Q}_{ijkl} = \mathbf{C}^0_{ijmn}(\mathbf{J}_{mnkl} - \mathbf{P}_{mrs\ell} \mathbf{C}^0_{rskl}), \quad \mathbf{P}_{ijkl} = \mathbf{S}^0_{ijmn}(\mathbf{J}_{mnkl} - \mathbf{Q}_{mrs\ell} \mathbf{S}^0_{rskl}),$$

$$\mathbf{\Theta}_{ijkl} = \mathbf{S}^*_{ijmn} \mathbf{P}_{mrs\ell} \mathbf{C}^0_{rskl}, \quad \mathbf{\Gamma}_{ijkl} = \mathbf{C}^*_{ijmn} \mathbf{Q}_{mrs\ell} \mathbf{C}^0_{rskl};$$

(2.5)
Utilizing these results, the tensors of compliance and stiffness contribution of the inclusion can be given by the formulas

\[ H = \frac{V_s}{V} [(S^* - S^0)^{-1} + Q]^{-1}, \quad N = \frac{V_s}{V} [(C^* - C^0)^{-1} + P]^{-1}. \]  

(2.6)

For a general ellipsoid, components \( H_{ijkl} \) and \( N_{ijkl} \) are expressed in terms of elliptic integrals. They reduce to elementary functions for the ellipsoid of revolution (spheroid); in this case tensors \( P, Q, H \) and \( N \) are derived in the text to follow (preliminary results on these tensors were reported by Sevostianov and Kachanov, 1999). Our analysis requires explicit analytic inversions of fourth rank tensors. Such inversions can be done by representing these tensors in terms of a certain “standard" tensorial basis \( T^{(1)}, \ldots, T^{(6)} \) (Kunin, 1983; see Appendix A):

\[ P = \sum_{k=1}^{6} p_k T^{(k)}, \quad Q = \sum_{k=1}^{6} q_k T^{(k)}, \quad H = \frac{V_s}{V} \sum_{k=1}^{6} h_k T^{(k)}, \quad N = \frac{V_s}{V} \sum_{k=1}^{6} n_k T^{(k)}, \]  

(2.7)

so that finding these tensors reduces to calculation of factors \( p_k, q_k, h_k \) and \( n_k \). Using the representations for tensors of elastic stiffness and compliance, Eshelby’s tensor and unit tensor in terms of this basis (Appendix A) yields the following relations for the coefficients \( p_i, q_i \):

\[ p_1 = \frac{1}{2G_0} [(1 - \kappa)f_0 + f_1], \quad p_2 = \frac{1}{2G_0} [(2 - \kappa)f_0 + f_1], \]
\[ p_3 = p_4 = - \frac{1}{G_0} f_1, \quad p_5 = \frac{1}{G_0} [1 - f_0 - 4f_1], \]
\[ p_6 = \frac{1}{G_0} [(1 - \kappa)(1 - 2f_0) + 2f_1], \]
\[ q_1 = G_0 [4\kappa - 1 - 2(3\kappa - 1)f_0 - 2f_1], \quad q_2 = 2G_0 [1 - (2 - \kappa)f_0 - f_1], \]
\[ q_3 = q_4 = 2G_0 [(2\kappa - 1)f_0 + 2f_1], \quad q_5 = 4G_0 [f_0 + 4f_1], \]
\[ q_6 = 8G_0 [\kappa f_0 - f_1]. \]  

(2.8)

The following notations are used hereafter:

\[ \kappa = \frac{1}{2(1 - \nu_0)}, \quad f_0 = \frac{\gamma^2 (1 - g)}{2(\gamma^2 - 1)}, \quad f_1 = \frac{\kappa \gamma^2}{4(\gamma^2 - 1)^2} [(2\gamma^2 + 1)g - 3], \]  

(2.9)

where the shape factor \( g \) is expressed in terms of the aspect ratio \( \gamma \) as follows:

\[ g(\gamma) = \begin{cases} \frac{1}{\gamma \sqrt{1 - \gamma^2}} \arctan \frac{1}{\gamma \sqrt{1 - \gamma^2}}, & \text{oblate shape} (\gamma < 1), \\ \frac{1}{2\gamma \sqrt{\gamma^2 - 1}} \ln \frac{\gamma + \sqrt{\gamma^2 - 1}}{\gamma - \sqrt{\gamma^2 - 1}}, & \text{prolate shape} (\gamma > 1). \end{cases} \]  

(2.10)
Factors $h_i$ entering the representation of the compliance contribution tensor $H$ of the inclusion in terms of the tensorial basis are given by

$$h_1 = \frac{1}{2 \Delta} \left[ K_1 + \frac{4}{3} G_1 + q_6 \right], \quad h_2 = \frac{1}{2 G_1 + q_2}, \quad h_5 = \frac{4}{4 G_1 + q_5},$$

$$h_3 = h_4 = -\frac{1}{\Delta} \left[ K_1 - \frac{2}{3} G_1 + q_3 \right], \quad h_6 = \frac{2}{\Delta} \left[ K_1 + \frac{1}{3} G_1 + q_1 \right], \quad (2.11)$$

where

$$K_1 = K_* K_0 / (K_* - K_0), \quad \delta G = G_* G_0 / (G_* - G_0),$$

$$\Delta = 2 \left[ 3 G_1 K_1 + K_1 (q_1 + q_6 - 2 q_3) + \frac{G_1}{3} (4 q_1 + q_6 + 4 q_3) + (q_1 q_6 - q_3^2) \right].$$

The stiffness contribution tensor $N$ of the inclusion has the following factors in this basis:

$$n_1 = \frac{1}{2 \Delta_1} \left[ \frac{\delta \lambda + \delta G}{\delta G (3 \delta \lambda + 2 \delta G)} + p_6 \right], \quad n_2 = \frac{2 \delta G}{1 + 2 p_2 \delta G},$$

$$n_3 = n_4 = -\frac{1}{\Delta_1} \left[ \frac{\delta \lambda}{2 \delta G (3 \delta \lambda + 2 \delta G)} + p_3 \right], \quad n_5 = \frac{4 \delta G}{1 + \delta G p_5},$$

$$n_6 = \frac{1}{\Delta_1} \left[ \frac{\delta \lambda + 2 \delta G}{2 \delta G (3 \delta \lambda + 2 \delta G)} + 2 p_1 \right], \quad (2.12)$$

where

$$\delta \lambda = \lambda_* - \lambda_0, \quad \delta G = G_* - G_0,$$

$$\Delta_1 = \frac{1}{2 \delta G (3 \delta \lambda + 2 \delta G)} [1 + (\delta \lambda + 2 \delta G) p_6 + 4 (\delta \lambda + \delta G) p_1 + 4 \delta \lambda p_5]$$

$$+ 2 p_1 p_6 - 2 p_3^2.$$
In the limiting case of a cavity, $\delta K = \delta G = 0$ and the analysis is simpler in terms of the compliance contribution tensor $H$. In this case,

$$h_1 = \frac{q_6}{4(q_1 q_6 - q_3^2)}, \quad h_2 = \frac{1}{q_2}, \quad h_3 = h_4 = -\frac{q_3}{2(q_1 q_6 - q_3^2)},$$

$$h_5 = \frac{4}{q_5}, \quad h_6 = \frac{q_1}{q_1 q_6 - q_3^2}. \quad (2.14)$$

In the case of overall transverse isotropy, the change in the elastic compliance tensor has the structure

$$S - S_0 = \frac{V_s}{V E_0} [W_1 II + W_2 tr J + W_3 (Inn + nnI)]$$

$$+ W_4 (J \cdot nn + nn \cdot J) + W_5 nnnn], \quad (2.15a)$$

where the coefficients $W_i$ are expressed in terms of the coefficients $h_i$ as follows:

$$W_1 = h_1 - h_2/2, \quad W_2 = h_2, \quad W_3 = 2h_3 + h_2 - 2h_1, \quad W_4 = h_5 - 2h_2,$$

$$W_5 = h_6 + h_1 + h_2/2 - 2h_3 - h_5. \quad (2.16a)$$

The dual expression for the change in the elastic stiffness due to the inclusion has the form

$$C - C_0 = \frac{V_s G_0}{V} [U_1 II + U_2 tr J + U_3 (Inn + nnI)]$$

$$+ U_4 (J \cdot nn + nn \cdot J) + U_5 nnnn], \quad (2.15b)$$

where coefficients $U_i$ are expressed in terms of coefficients $n_i$ as follows:

$$U_1 = n_1 - n_2/2, \quad U_2 = n_2, \quad U_3 = 2n_3 + n_2 - 2n_1, \quad U_4 = n_5 - 2n_2,$$

$$U_5 = n_6 + n_1 + n_2/2 - 2n_3 - n_5. \quad (2.16b)$$

We now consider a solid with multiple inclusions in the non-interaction approximation. This approximation is of fundamental importance: besides being rigorous at small concentration of inclusions, it is the basic building block for various averaging schemes. In this approximation, each inclusion is placed in the remotely applied stress $\sigma$ and is not influenced by neighboring inclusions. Then

$$\Delta H_{ij} = H^{(m)}_{ijkl} \sigma_{kl} \quad (2.17)$$

and summation over inclusions may be replaced by integration over orientations, if computationally convenient.

**3. Approximate representation of the compliance/stiffness contribution tensors in terms of a second rank symmetric tensor**

Establishing the sought cross-property correlations crucially depends on the possibility to express, with sufficient accuracy, the compliance (or stiffness) contribution
tensor of an inclusion in terms of a certain second rank tensor. The following two issues should be addressed in this context.

(A) For a solid with one inclusion, we identify the inclusion shapes for which the characterization by $H$ (or $N$) tensor can be reduced, with sufficient accuracy, to one in terms of a certain second rank symmetric tensor $\Omega$:

$$H = \frac{1}{E_0} \frac{V}{V} \begin{bmatrix} B_1 I + B_2 J + B_3 (\Omega I + I \Omega) + B_4 (\Omega \cdot J + J \cdot \Omega) \end{bmatrix}$$  \hspace{1cm} (3.1a)$$

$$N = G_0 \frac{V}{V} \begin{bmatrix} D_1 I + D_2 J + D_3 (\Omega I + I \Omega) + D_4 (\Omega \cdot J + J \cdot \Omega) \end{bmatrix}$$  \hspace{1cm} (3.1b)$$

where $B_i$ and $D_i$ are scalar coefficients that depend on the inclusion shape and on the matrix–inclusion elastic contrast. The “isotropic terms” in (3.1) are expressed in terms of the second rank and fourth rank unit tensors ($I_{ij} = \delta_{ij}$ and $2J_{ijkl} = \delta_{ik}\delta_{jl} + \delta_{ij}\delta_{kl}$) and, thus, do not depend on the inclusion orientation.

(B) For a solid with many inclusions (analyzed in the framework of the non-interaction approximation) a similar structure should be established for the sum $\sum H^{(k)}$ (or $\sum N^{(k)}$).

Aside from being a key point in deriving the cross-property correlation, this simplification—representation of $H$ (or $N$) in terms of a certain second rank symmetric tensor—has important implications for the overall elastic anisotropy, as discussed at the end of the present section.

In the case of axial symmetry of the inclusion shape (spheroid, for example), representation (3.1) implies the following restrictions on the coefficients $h_i$:

$$h_6 + h_1 + h_2/2 - 2h_3 - h_5 = 0.$$  \hspace{1cm} (3.2)$$

With the trivial exception of a sphere, representations (3.1) do not hold exactly. But condition (3.2) is satisfied, with good accuracy, for spheroids within several ranges of parameters that, being sufficiently wide, are relevant for realistic matrix composites.

Indeed, let us discuss, for certainty, the compliance contribution tensor $H$ (similar considerations apply to the stiffness contribution tensor $N$). One can construct a fictitious compliance contribution tensor $\hat{H}$, with coefficients $\hat{h}_i$ in the tensorial basis that are obtained from $h_i$ by multiplication of $h_i$ by either $(1 + \delta)$ or $(1 - \delta)$, and choose $\delta$ in such a way that condition (3.2) is satisfied exactly for $\hat{h}_i$:

$$\hat{h}_1 = h_1(1 - \delta \text{sign} h_1), \quad \hat{h}_3 = h_3(1 + \delta \text{sign} h_3),$$

$$\hat{h}_2 = h_2(1 - \delta \text{sign} h_2), \quad \hat{h}_5 = h_5(1 + \delta \text{sign} h_5),$$

$$\hat{h}_6 = h_6(1 - \delta \text{sign} h_6),$$  \hspace{1cm} (3.3)$$
where
\[ \delta = \frac{h_6 + h_1 + h_2 - 2h_3 - h_5}{|h_6| + |h_1| + |h_2|/2 + 2|h_3| + |h_5|}. \] (3.4)

Then we find that the error of this approximation, as estimated by the norm \( \max_{ijkl,Hijkl \neq 0}|(Hijkl - \hat{H}ijkl)|/|Hijkl| \), is equal to \( |\delta| \). The choice of this norm, as the measure of accuracy of representation (3.1), corresponds to the requirement that strain responses to all stress states of the actual and of the fictitious inclusions are close if the norm is small.

Thus, we obtain the following expressions for the coefficients \( B_i \):
\[ B_1 = E_0(\hat{h}_1 - \hat{h}_2/2), \quad B_2 = E_0\hat{h}_2, \]
\[ B_3 = E_0(2\hat{h}_3 + \hat{h}_2 - 2\hat{h}_1), \quad B_4 = E_0(\hat{h}_5 - 2\hat{h}_2). \] (3.5a)

The coefficients \( D_i \) for the stiffness contribution tensor \( N \) can be obtained similarly:
\[ D_1 = (\hat{n}_1 - \hat{n}_2/2)/G_0, \quad D_2 = \hat{n}_2/G_0, \]
\[ D_3 = (2\hat{n}_3 + \hat{n}_2 - 2\hat{n}_1)/G_0, \quad D_4 = (\hat{n}_5 - 2\hat{n}_2)/G_0. \] (3.5b)

Figs. 1 and 2 illustrate these coefficients as functions of the inclusion’s aspect ratio and of the matrix/inclusion elastic contrast.

The accuracy of approximation (3.5) depends on the following three factors: (1) elastic contrast between the matrix and the inclusion’s material; (2) spheroid’s aspect ratio \( b_{CR} = a_3/a \) and (3) Poisson’s ratios \( v_0 \) and \( v_* \). The accuracy issue is discussed in Section 7, where it is illustrated by “accuracy maps”.

For a solid with many inclusions, we seek to approximate the sum (over all inclusions) by the expressions
\[ \sum H^{(k)} = \frac{1}{E_0} \left[ cb_1 II + cb_2 J + b_3(\omega I + I\omega) + b_4(\omega \cdot J + J \cdot \omega) \right], \] (3.6a)
\[ \sum N^{(k)} = G_0 \left[ cd_1 II + cd_2 J + d_3(\omega I + I\omega) + d_4(\omega \cdot J + J \cdot \omega) \right], \] (3.6b)

where \( c \) is the volume concentration of inclusions and \( b_i, d_i \) are scalar coefficients that depend on the average inclusion shapes, as well as on the matrix constants \( E_0 \) and \( v_0 \).

Note that representation (3.1) for one inclusion constitutes a necessary, but not a sufficient condition for representation (3.6) to hold (with an important exception of the case when all the inclusions have identical shapes). This is due to the fact that, for
Fig. 1. Factors entering the approximate expression (3.1a) for the compliance contribution tensor $H$ as functions of the inclusion’s aspect ratio $\gamma$. Poisson’s ratios $\nu_0 = \nu_0^* = 0.3$. Curves 1 to 6 correspond to ratios $E_*/E_0 = 0.01, 0.1, 0.33, 3.0, 10$ and 100. For oblate shapes ($\gamma < 1$) factors $B_i$ enter in product with inclusion volume $V_*$, to avoid degeneracy for small $\gamma$, and the logarithmic scale is inverted for a convenient comparison with pore shapes.

Mixtures of diverse shapes, coefficients $B_i, D_i$ entering (3.1), are different for different inclusions. The analysis of Section 5 shows that, nevertheless, representation (3.6) holds for a wide range of realistic microstructures.
Fig. 2. Factors entering the approximate expression (3.1b) for the stiffness contribution tensor $N$ as functions of the inclusion’s aspect ratio $\gamma$. Poisson’s ratios $\nu_0 = \nu_1 = 0.3$. Curves 1 to 6 correspond to ratios $E_*/E_0 = 0.01, 0.1, 0.33, 3.0, 10$ and $100$. For oblate shapes ($\gamma < 1$) factors $D_i$ enter in product with inclusion volume $V_*$, to avoid degeneracy for small $\gamma$, and the logarithmic scale is inverted for a convenient comparison with pore shapes.

**Remark.** Aside from being a key point in establishing the cross-property correlations, representation (3.6) (when it is possible) has far reaching implications, as follows:

1. It implies that a solid with inclusions is approximately orthotropic (orthotropy being coaxial with the principal axes of $\omega$). We emphasize that the orthotropy holds
for any orientational and aspect ratio distributions of inclusions, including cases when
the orthotropic symmetry does not seem to agree with intuition (like several families
of parallel inclusions inclined at arbitrary angles to each other).

2. Moreover, the orthotropy due to inclusions is of a special, simplified type. This
is due to the fact that the effective compliance tensor $S$ can be expressed in terms of
a symmetric second rank tensor $\omega$.

4. Resistivity contribution tensor of an inclusion

We consider now the thermal conductivity problem (that is mathematically equivalent
to the electric conductivity problem). We first consider a reference volume $V$ of an
infinite three-dimensional solid (with the isotropic thermal conductivity $k_0$) containing
an ellipsoidal inclusion with the isotropic thermal conductivity $k_*$ (limiting cases $k_*=0$
and $\infty$ correspond to an insulator and a superconductor). Assuming a linear conduction
law (linear relation between the far-field temperature gradient $G$ and the heat flux
vector $U$ per volume $V$), we then have a linear relation for the change due to the
inclusion:

$$\Delta G = \frac{1}{V} H^R \cdot U,$$

(4.1)

where symmetric second rank tensor $H^R$ can be called the resistivity contribution
tensor of an inclusion; superscript R indicates resistivity, in contrast with $H$ for
the elasticity. (Formula (4.1) can be interpreted as the change in temperature gra-
dient that is required to maintain the same heat flux after the inclusion has been
introduced.)

Utilizing the same approach as the one for the elastic problem, tensor $H^R$ can be
expressed in terms of Eshelby’s tensor for conductivity $s^K$:

$$H^R = V_*/V k_0 \left( \frac{k_0}{k_* - k_0} I - s^K \right)^{-1}$$

(4.2)

In the case of the spheroidal pore, tensor $s^K$ was derived by Levin et al. (2000) as a
part of a more general piezoelectric problem:

$$s^K = k_0 [ f_0(\gamma)(I - nn) + (1 - 2 f_0(\gamma))nn ],$$

(4.3)

where $f_0(\gamma)$ is given in (2.9). Substituting this result into (4.2) yields the following
expression for $H^R$:

$$H^R = V_*/k_0 \{ A_1 I + A_2 nn \},$$

(4.4)

where factors $A_1$ and $A_2$ (illustrated in Fig. 3) are as follows:

$$A_1 = \frac{k_0 - k_*}{k_0 + (k_* - k_0) f_0(\gamma)}.$$
Fig. 3. Factors entering the approximate expression (4.4) for the conductivity contribution tensor $H^R$ as functions of the inclusion’s aspect ratio $\gamma$. Curves 1–6 correspond to ratios $k_*/k_0 = 0.01, 0.1, 0.33, 3.0, 10$ and 100. For oblate shapes ($\gamma < 1$) factors $A_i$ enter in product with inclusion volume $V_*$, to avoid degeneracy for small $\gamma$, and the logarithmic scale is inverted for a convenient comparison with pore shapes. The data for $A_2$ is given by two sets of curves due to a high sensitivity to the conductivity contrast $k_*/k_0$.

$$A_2 = \frac{(k_0 - k_*)^2(1 + 3f_0(\gamma))}{[k_* - 2(k_* - k_0)f_0(\gamma)][k_0 + (k_* - k_0)f_0(\gamma)]}. \quad (4.5)$$

For a sphere, $\gamma = 1$ and $f_0(\gamma) = 1/3$.
For a cylinder, $\gamma \to \infty$ and $f_0(\gamma) = 1/2$.
For a crack ($\gamma \to 0$ and $V_*\gamma \to (4/3)\pi a^3$):

$$H^R = 8a^3 \frac{1}{3} \frac{1}{k_0} nn. \quad (4.6)$$

For a superconducting disk, $\gamma \to 0$ and $k_* = \infty$, $V_*\gamma \to (4/3)\pi a^3$:

$$H^R = -\frac{16a^3}{3V} \frac{1}{k_0} (I - nn). \quad (4.7)$$

**Remark.** An important observation relevant for many realistic microstructures (and similar to the one concerning the effective elasticity) is that $H^R$ for a crack can be used, with good accuracy, for strongly oblate pores (up to $\gamma = 0.15$). Indeed, Fig. 3 shows that $A_1 \ll A_2$ for the strongly oblate shapes, whereas $A_2$ is almost flat at small $\gamma$. 
5. Explicit cross-property correlations for two-phase matrix composites

For a solid with many inclusions, we obtain, utilizing relations (3.8) and (4.5), the following effective compliances, stiffnesses and conductivities:

\[
S = S_0 + \frac{1}{E_0} \frac{1}{V} \left[ \sum_i (V_i B_1(i)) + J \sum_i (V_i B_2(i)) \right] + \left[ \sum_i (V_i B_3(i)) \right]_{\text{symm}} + \left[ J \cdot \sum_i (V_i B_4(i)) \right]_{\text{symm}},
\]

\[
C = C_0 + \frac{G_0}{V} \left[ \sum_i (V_i D_1(i)) + J \sum_i (V_i D_2(i)) \right] + \left[ \sum_i (V_i D_3(i)) \right]_{\text{symm}} + \left[ J \cdot \sum_i (V_i D_4(i)) \right]_{\text{symm}},
\]

\[
K^{-1} - K_0^{-1} = \frac{1}{V k_0} \left\{ \sum_i (V_i A_1(i)) + \sum_i (V_i A_2(i)) \right\},
\]

where coefficients \(B_i\) and \(A_i\) are given by (3.5) and (4.5), respectively, and the subscript “symm” refers to the symmetrization appropriate for the elasticity tensors. The general structure of these formulas (as well as the logic of the derivation to follow) is similar to the one for the porous materials (Kachanov et al., 2001).

These formulae apply to an arbitrary mixture of inclusions of diverse aspect ratios and orientations and contain factors \(A_i, B_i\) and \(D_i\) that depend on the inclusion shapes and, also, on the inclusion conductivity and elastic moduli. Since these factors are different for different inclusions, tensors \(\sum_i (V_i B_{i\text{nn}})\), \(\sum_i (V_i D_{i\text{nn}})\) and \(\sum_i (V_i A_{i\text{nn}})\) entering \(S\), \(C\) and conductivity tensor \(K\), respectively, cannot, generally, be expressed in terms of each other and may not even be coaxial.

However, if the inclusions’ aspect ratios are not correlated with either orientations of the inclusions or their volumes (note that volumes and orientations may be correlated), coefficients \(D_i, B_i\) and \(A_i\) can be replaced by their averages and taken out of the summation signs (if all the inclusions have the same orientation \(n\), this requirement reduces to the condition that the distributions over shapes and over volumes of the inclusions are uncorrelated). This important case appears to be relevant for realistic microstructures.

Remark. The requirement of absence of the mentioned correlations may actually be substantially relaxed. As shown by Kachanov et al. (2001), for porous materials, the cross-property correlation can be extended, in many cases, to microstructures comprising two or three families of pores with distinctly different pore statistics. This result can be directly extended to the matrix composites.
Then three tensors $S$, $C$ and $K$ are expressed in terms of the same second rank symmetric tensor (“inclusions’ concentration tensor”):

$$
\omega = \frac{1}{V} \sum_k (V_\ast mn)^{(k)}.
$$

(5.3)

Note that its trace $tr \omega = (1/V) \sum V_\ast$ is the volume fraction of inclusions $c$. Thus,

$$
S = S_0 + \frac{1}{E_0} c(b_1 II + b_2 J) + \frac{1}{E_0} [b_3(\omega I + I\omega) + b_4(\omega \cdot J + J \cdot \omega)],
$$

(5.4a)

$$
C = C_0 + G_0 c(d_1 II + d_2 J) + G_0[d_3(\omega I + I\omega) + d_4(\omega \cdot J + J \cdot \omega)],
$$

(5.4b)

$$
K^{-1} - K_0^{-1} = \frac{1}{k_0} \{a_1 c I + a_2 \omega\}.
$$

(5.5)

Coefficients $b_i$, $d_i$ and $a_i$—average shape factors for the elasticity problem and for the conductivity problem, respectively—are averages (over all the cavities) of coefficients $B_i$, $D_i$ and $A_i$:

$$
b_i = \int_0^\infty B_i(\gamma) f(\gamma) d\gamma, \quad d_i = \int_0^\infty D_i(\gamma) f(\gamma) d\gamma, \quad a_i = \int_0^\infty A_i(\gamma) f(\gamma) d\gamma,
$$

(5.6)

where $f(\gamma)$ is the shape distribution density. Functions $B_i(\gamma)$, $D_i(\gamma)$ and $A_i(\gamma)$, given by (3.5) and (4.5), are illustrated in Figs. 1–3.

**Remark.** The possibility to express elasticity tensors $S$ and $C$ in terms of the symmetric second rank tensor $\omega$ has interesting physical implications. Besides implying the overall elastic orthotropy for any orientational distribution of inclusions, it also implies that the orthotropic elastic tensors are coaxial with $\omega$ and, therefore, are coaxial with the overall conductivity tensor $K$. The accuracy of these statements is determined by the accuracy of representation of $S$ and $C$ in terms $\omega$; it is discussed in Section 7.

We now return to cross-property correlations. Expressing the inclusion concentration tensor $\omega$ in terms of $K$ from (5.5) and substituting it into (5.4) yields a cross-property correlation—a closed form expression of the effective compliance tensor in terms of the effective conductivity tensor:

$$
E_0(S - S_0) = \left[ \frac{b_1 a_2 - b_3 a_1}{a_2(a_2 + 3a_1)} II + \frac{b_2 a_2 - b_4 a_1}{a_2(a_2 + 3a_1)} J \right] [tr(k_0 K^{-1}) - 3]
$$

$$
+ \frac{b_3 k_0}{a_2} [(K^{-1} - I) I + I (K^{-1} - I)]
$$

$$
+ \frac{b_4 k_0}{a_2} [(K^{-1} - I) \cdot J + J \cdot (K^{-1} - I)].
$$

(5.7a)
and a similar expression for the effective stiffness tensor:

\[
\frac{(C - C_0)}{G_0} = \left[ \frac{d_1 a_2 - d_3 a_1}{a_2(a_2 + 3a_1)} \mathbf{II} + \frac{d_2 a_2 - d_4 a_1}{a_2(a_2 + 3a_1)} \mathbf{J} \right] [\text{tr}(k_0 K^{-1}) - 3]
\]

\[
+ \frac{d_3 k_0}{a_2} [(K^{-1} - I) I + I(K^{-1} - I)]
\]

\[
+ \frac{d_4 k_0}{a_2} [(K^{-1} - I) \cdot J + J \cdot (K^{-1} - I)].
\] (5.7b)

Both of expressions (5.7) are approximate (due to the approximate character of representations (3.8) of the elasticity tensor in terms of a second rank tensor). Of these two expressions, (5.7b) is more accurate, especially in the case of low compressibility of the matrix material (see discussion of Section 7).

Each of the two forms of the derived cross-property correlations, (5.7a) and (5.7b), contains \textit{four shape factors}—combinations of \(a_i, b_i\) and \(d_i\)—that depend on the average pore shapes. Their presence reflects the fact that the influence of inclusion shapes on the elastic and on the conductive effective properties is somewhat different (otherwise, the cross-property correlations would have been inclusion shape-independent).

We emphasize that the derived cross-property correlations cover all inclusion shapes (including \textit{mixtures} of diverse shapes) and orientational distributions of inclusions in a unified way. They contain elastic constants and conductivities of the constituents and average shapes of the inclusions (and do not contain any adjustable parameters).

The derived cross-property correlations express the change in the elastic compliance/stiffness tensor due to inclusions as a linear function of the change in the conductivity tensor. For applications, it is convenient to have these correlations in components. They are as follows (in the principal axes of the effective conductivity tensor that, as mentioned above, coincide with the principal axes of the elastic orthotropy):

\[
\frac{C_{1111} - C_{1111}^0}{G_0} = \frac{2(d_1 + d_2)a_2 + 2(d_3 + d_4)(a_2 + 2a_1)}{a_2(a_2 + 3a_1)} \frac{k_0 - k_{11}}{k_{11}} 
\]

\[
+ \frac{(d_1 + d_2)a_2 - (d_3 + d_4)a_1}{a_2(a_2 + 3a_1)} \frac{k_0 - k_{33}}{k_{33}},
\]

\[
\frac{C_{3333} - C_{3333}^0}{G_0} = 2 \frac{(d_1 + d_2)a_2 - (d_3 + d_4)a_1}{a_2(a_2 + 3a_1)} \frac{k_0 - k_{11}}{k_{11}}
\]

\[
+ \frac{(d_1 + d_2)a_2 + (d_3 + d_4)(2a_2 + 5a_1)}{a_2(a_2 + 3a_1)} \frac{k_0 - k_{33}}{k_{33}},
\]

\[
\frac{C_{1122} - C_{1122}^0}{G_0} = \frac{2(d_1 + d_2)a_2 + 4d_3 a_1}{a_2(a_2 + 3a_1)} \frac{k_0 - k_{11}}{k_{11}} + \frac{d_1 a_2 - d_2 a_1}{a_2(a_2 + 3a_1)} \frac{k_0 - k_{33}}{k_{33}},
\]
\[
\frac{C_{1133} - C^0_{1133}}{G_0} = \frac{2d_1a_2 + d_3(a_2 + a_1) k_0 - k_{11}}{a_2(a_2 + 3a_1)} \frac{k_0 - k_{33}}{k_{33}} + \frac{(d_1 + d_3)a_2 + 2d_3a_1}{a_2(a_2 + 3a_1)} \frac{k_0 - k_{33}}{k_{33}},
\]
\[
\frac{C_{1313} - C^0_{1313}}{G_0} = \frac{2d_2a_2 + d_4(a_2 - a_1) k_0 - k_{11}}{a_2(a_2 + 3a_1)} \frac{k_0 - k_{33}}{k_{33}} + \frac{d_2a_2 + d_4(a_2 + 2a_1)}{a_2(a_2 + 3a_1)} \frac{k_0 - k_{33}}{k_{33}}.
\]

Note that the coefficients interrelating the conductivity changes \(k_0/k_{11}, k_0/k_{33}\) and the elastic moduli changes can actually be expressed in terms of four coefficients entering the tensorial form (5.7b).

The utility of the explicit cross-property correlation (5.7) can be viewed as follows. If the effective conductivity tensor \(K\) is known, then the only additional information needed to find the full set of anisotropic effective elastic constants is the knowledge of average inclusion shapes—factors \(b_i\) (or \(d_i\)) and \(a_i\) and not the orientational distribution. Without the cross-property correlation, tensors \(S\) and \(C\) can be expressed in terms of inclusion concentration tensor \(\omega\) (formulae (5.4)). However, its knowledge requires a rather detailed information (of the orientational character) on the microstructure and may not be readily available. Utilization of the cross-property correlation makes the knowledge of \(\omega\) unnecessary.

5.1. The case of overall isotropy

If the composite material is isotropic (inclusions are either spherical or randomly oriented), tensor \(\omega\) is isotropic (\(\omega = cI\)) and the cross-property correlation (5.7) takes the following form, that contains only two shape factors—coefficients at \(II\) and \(J\):

\[
S = S_0 + \frac{k_0 - k}{kE_0} \left( \frac{b_1 + b_3}{3a_1 + a_2} II + \frac{b_2 + b_4}{3a_1 + a_2} J \right),
\]

\[
C = C_0 + \frac{G_0(k_0 - k)}{k} \left( \frac{d_1 + d_3}{3a_1 + a_2} II + \frac{d_2 + d_4}{3a_1 + a_2} J \right).
\]

In particular, the effective Young’s modulus and Poisson’s ratio are expressed (using (5.9a)) in terms of the effective conductivity \(k\) as follows:

\[
\begin{align*}
\frac{E_0 - E}{E} &= \frac{3}{3a_1 + a_2} \frac{b_1 + b_2 + b_3 + b_4}{k} k_0 - k, \\
\nu &= \frac{\nu_0 k(3a_1 + a_2) + 3(k_0 - k)(b_1 + b_3)}{[k(3a_1 + a_2) + 3(k_0 - k)(b_1 + b_2 + b_3 + b_4)]},
\end{align*}
\]

We note that the overall isotropy takes place in one of the two cases: (A) spherical inclusions and (B) randomly oriented non-spherical inclusions. Whereas in case (A) the cross-property correlations (5.9) are exact, in case (B) they are approximate, since they are based on the approximate representations (3.2) of tensors \(H\) and \(N\) in terms of a second rank tensor. However, in case (B), the exact cross-property correlations can be derived independently (without using approximation (3.2)). Indeed, in this case,
the exact representation (2.15a) of the effective elastic compliance tensor is

\[
E_0(S - S_0) = c(w_1 + w_3/3 + w_5/15)H + c(w_2 + w_4/3 + w_5/15)J,
\]

(5.11)

where \(w_i\) are related to \(W_i\) for the individual inclusions (given by (2.16a)) by formulas analogous to (5.6). The effective conductivity in the case of isotropy takes the form

\[
k = k_0/(1 - a_1 c - a_2 c/3).
\]

(5.12)

Solving for the volume fraction \(c\) from (5.12) and substituting into (5.11) yields the exact cross-property correlation

\[
E_0(S - S_0) = \frac{3(k_0 - k)}{k} \left[ \frac{w_1 + w_3/3 + w_5/15}{3a_1 + a_2}H + \frac{w_2 + w_4/3 + w_5/15}{3a_1 + a_2}J \right].
\]

(5.13)

In particular, the effective Young’s modulus and Poisson’s ratio are exactly expressed in terms of the effective conductivity \(k\) as follows:

\[
\frac{E_0 - E}{E} = \frac{3(w_1 + w_2) + w_3 + w_4 + 2w_5/5}{3a_1 + a_2} \frac{k_0 - k}{k},
\]

\[
v = \frac{\nu_0 k(3a_1 + a_2) - (k_0 - k)(3w_1 + w_3 + w_5/5)}{[k(3a_1 + a_2) + (k_0 - k)(3w_1 + 3w_2 + w_3 + w_4 + 2w_5/5)]},
\]

(5.14)

6. Applications

We now specialize the derived cross-property correlations for several material systems. The utility of the obtained correlations is demonstrated by estimating the full set of anisotropic effective elastic constants from the measured effective conductivities, without direct knowledge of the orientational distribution of inclusions.

6.1. Homogeneous material with microcracks

In the case of isotropy (random crack orientations), the explicit cross-property correlations were given by Bristow (1960); in the general anisotropic case, such correlations were derived by Kachanov et al. (2001). In the latter work, the analysis was done in terms of compliance contribution tensors (\(H\)-tensors) of cracks. The accuracy of such formulation worsens noticeably for materials of low compressibility (\(\nu_0 > 0.4\)). Here, we supplement the earlier analysis by a dual formulation in terms of the stiffness contribution tensors (\(N\)-tensors). While the formulation in terms of \(N\)-tensors leads to somewhat lengthier cross-property expressions (as compared to the ones obtained by using \(H\)-tensors), its accuracy is, generally, better and is particularly good for matrices of low compressibility (becoming exact in the limit \(\nu_0 \rightarrow 0.5\)). In the case of cracks, both the effective conductivities and the approximate representation of the elastic properties are given in terms of a symmetric second rank crack density...
(introduced by Kachanov, 1980 and, in a different form, by Vakulenko and Kachanov, 1971). Its first invariant $\text{tr} \mathbf{a}$ coincides with the conventional scalar crack density $\rho = (1/V) \sum a^3$ introduced by Bristow (1960). In terms of $\mathbf{a}$, the effective conductivity tensor is expressed exactly:

$$ K = k_0(I + 8\mathbf{a}/3)^{-1} $$

and the effective compliance tensor $\mathbf{S}$ and the effective stiffness tensor $\mathbf{C}$ are expressed approximately as follows:

$$ \mathbf{S} = \mathbf{S}_0 + \frac{16(1 - v_0^2)}{3(2 - v_0)E_0} (\mathbf{a} \cdot \mathbf{J} + \mathbf{J} \cdot \mathbf{a}), $$

$$ \mathbf{C} = \mathbf{C}_0 - \frac{64G_0(1 - v_0)}{3V(4 - 9v_0 + 8v_0^2)} \left[ \frac{v_0^2(2 - 5v_0 + 6v_0^2 - 2v_0^3)}{(1 - 2v_0)^2} \right] \mathbf{II} $$

$$ + \frac{2(1 - v_0)(1 - 2v_0 + v_0^2 + v_0^3) + v_0^2(2 - 5v_0 + 6v_0^2 - 2v_0^3)}{(1 - 2v_0)^2} \left( (\mathbf{I}\mathbf{a} + \mathbf{a}\mathbf{I}) \right. $$

$$ + \frac{2(1 - 2v_0 + v_0^2 + v_0^3)}{2 - v_0} \left. \mathbf{J} \cdot \mathbf{a} + \mathbf{a} \cdot \mathbf{J} \right] . $$

Although formula (6.3a) for $\mathbf{S}$ is simpler, expression (6.3b) for $\mathbf{C}$ has a substantially better accuracy (see Section 7). Expressing now $\mathbf{a}$ in terms of $\mathbf{K} - \mathbf{K}_0$ from (6.2) and substituting into (6.3) yields the explicit cross-property correlations in two dual forms (of which the second one, albeit lengthier, has better accuracy, particularly at high $v_0$):

$$ \mathbf{S} = \mathbf{S}_0 + \frac{2(1 - v_0^2)}{(2 - v_0)E_0} \left[ (k_0\mathbf{K}^{-1} - \mathbf{I}) \cdot \mathbf{J} + \mathbf{J} \cdot (k_0\mathbf{K}^{-1} - \mathbf{I}) \right], $$

$$ \mathbf{C} = \mathbf{C}_0 - \frac{8G_0(1 - v_0)}{k_0(4 - 9v_0 + 8v_0^2)} \left[ \frac{v_0^2(2 - 5v_0 + 6v_0^2 - 2v_0^3)}{(1 - 2v_0)^2} \right] \text{tr}(k_0\mathbf{K}^{-1} - \mathbf{I}) \mathbf{II} $$

$$ + \frac{2 - 6v_0 + 8v_0^2 - 5v_0^3 + 4v_0^4 - 2v_0^5}{(1 - 2v_0)^2} \left( (k_0\mathbf{K}^{-1} - \mathbf{I}) + (k_0\mathbf{K}^{-1} - \mathbf{I}) \mathbf{I} \right) $$

$$ + \frac{2(1 - 2v_0 + v_0^2 + v_0^3)}{2 - v_0} \left( (k_0\mathbf{K}^{-1} - \mathbf{I}) + (k_0\mathbf{K}^{-1} - \mathbf{I}) \cdot \mathbf{J} \right) . $$

A simpler, but somewhat less accurate cross-property correlation (6.4a) implies a very simple one-to-one correspondence between the effective Young’s moduli $E_i$ and the
principal conductivities $k_i$ in the same directions:
\[
\frac{E_i}{E_0 - E_i} = \frac{2 - \nu_0}{2(1 - \nu_0^2)} \frac{k_i}{k_0 - k_i}.
\] (6.5)

More accurate cross-property correlations (6.4b) imply the following, somewhat lengthier relations:
\[
(C_{1111} - C_{1111}^0)/G_0 = 2(q_1 + q_2 + q_3)(k_{11} - k_0)/k_{11} + q_1(k_{33} - k_0)/k_{33},
\]
\[
(C_{3333} - C_{3333}^0)/G_0 = 2q_1(k_{11} - k_0)/k_{11} + (q_1 + 2q_2 + 2q_3)(k_{33} - k_0)/k_{33},
\]
\[
(C_{1122} - C_{1122}^0)/G_0 = 2(q_1 + q_3)(k_{11} - k_0)/k_{11} + q_1(k_{33} - k_0)/k_{33},
\]
\[
(C_{1133} - C_{1133}^0)/G_0 = (2q_1 + q_3)(k_{11} - k_0)/k_{11} + (q_1 + q_3)(k_{33} - k_0)/k_{33},
\]
\[
(C_{1313} - C_{1313}^0)/G_0 = q_3(k_{11} - k_0)/k_{11} + q_3(k_{33} - k_0)/k_{33},
\] (6.6)

where
\[
q_1 = \frac{-8v_0^3(1 - \nu_0)(2 - 5\nu_0 + 6\nu_0^2 - 2\nu_0^3)}{(4 - 9\nu_0 + 8\nu_0^2)(1 - 2\nu_0)^2},
\]
\[
q_2 = \frac{-16\nu_0(1 - \nu_0)(2 - 4\nu_0 + 3\nu_0^2)}{(4 - 9\nu_0 + 8\nu_0^2)(1 - 2\nu_0)},
\]
\[
q_3 = \frac{-8(1 - \nu_0)(1 - 2\nu_0 + \nu_0^2 + \nu_0^3)}{(4 - 9\nu_0 + 8\nu_0^2)(2 - \nu_0)}.
\] (6.7)

Note that, since formulas (6.3a) and (6.3b) utilized in the derivation of (6.4) are both approximate, results (6.4a) and (6.4b) do not follow from each other.

**Remark.** A physically important observation is that, as far as the effective conductivities, effective compliances and the cross-property correlations are concerned, strongly oblate pores can be replaced by cracks (Kachanov et al., 2001). Thus, all the relations derived in the present subsection, apply to materials with pores having aspect ratios $\gamma < 0.15$.

### 6.2. Rigid discs in a matrix of very low compressibility (reinforced plastics)

We introduce a parameter
\[
\zeta = (1 - 2\nu_0)/\pi\gamma,
\] (6.8)

that is a ratio of two small values (the matrix compressibility and disc’s aspect ratio $\gamma \ll 1$). For certainty, we assume in this section that $\zeta \ll 1$. This covers, for example, the case of $\nu_0 = 0.49$ (polyurethane or polybutadiene elastomer matrix) and $\gamma = 0.1$. 
A sufficiently accurate representation of the effective elastic properties in terms of symmetric second rank tensor $\omega$ can be given in this case if the stiffness contribution tensor $N$ (rather than compliance contribution tensor $H$) is used.

Retaining terms of the first order in $\zeta$ only yields the following results for the effective stiffness tensor:

$$C = C_0 + G_0 \frac{8(1 - v_0)}{3(7 - 8v_0)} \left[ \frac{4\zeta - (7 - 8v_0)(1 - 4v_0)}{3 - 4v_0} \right] H + 16J + G_0 \frac{8(1 - v_0)}{3(7 - 8v_0)} \left[ \frac{4\pi\gamma(7 - 8v_0) + 4(11 - 16v_0)}{3 - 4v_0} (\mathbf{aI + Ia} - 32(\mathbf{aJ + J.a})) \right].$$

(6.9)

For the effective conductivity, we have

$$\sum H^R = \frac{16(k^* - k_0)}{3k_0} \{[(\pi\gamma/(4k_0 + \pi\gamma k^*))](c\mathbf{I - a}) + (1/4k_0)\mathbf{a}\}. \quad (6.10)$$

In the case of highly conducting disks ($k^* \gg k_0$) that is relevant, for example, for metal reinforced plastics, this formula simplifies to

$$\sum H^R = -16k_0(\mathbf{I - a})/3. \quad (6.11)$$

Both $C$ and $K$ are thus given in terms of the inclusion concentration tensor $\mathbf{a}$ that coincides in the considered case with the crack density tensor (6.1). Expressing $\mathbf{a}$ from (6.11) and substituting it into (6.9) yields the explicit cross-property correlation:

$$\frac{C_{1111} - C_{1111}^0}{C_{1111}^0} = \frac{C_{3333} - C_{3333}^0}{C_{3333}^0} = \frac{(1 - 4v_0)\pi\gamma}{4(3 - 4v_0)} [(k_0 - k_{11})/k_{11} + (k_0 - k_{33})/2k_{33}],$$

$$\frac{C_{1122} - C_{1122}^0}{C_{1122}^0} = \frac{C_{1133} - C_{1133}^0}{C_{1133}^0} = \frac{(1 - 4v_0)(1 - v_0)\pi\gamma}{8v_0(3 - 4v_0)} [(k_0 - k_{11})/k_{11} + (k_0 - k_{33})/2k_{33}],$$

$$\frac{C_{1313} - C_{1313}^0}{C_{1313}^0} = \frac{8(1 - v_0)}{(7 - 8v_0)} [(k_0 - k_{11})/k_{11} - (k_0 - k_{33})/k_{33}]. \quad (6.12)$$

6.3. Multiphase materials: fiber reinforced composite with cracks

The examples analyzed above deal with the two-phase materials. In fact, our approach can be extended, in a straightforward way, to a matrix containing $N$ different phases, provided $N - 1$ of them have known structures (known shapes, orientations and concentrations).

A particularly interesting case of this sort is a material reinforced by fibers (of a given concentration and orientational distribution) and containing a certain (unknown) field of
cracks. We consider here the case of parallel fibers in the direction $n$ (unidirectionally reinforced composite).

The effective conductivity tensor and the effective stiffness tensor are as follows:

\[ k_0 K^{-1} = I + \frac{2c_f}{3} \frac{k_0 - k_f}{k_0 + k_f} \left( I + \frac{k_0 - k_f}{k_f} nn \right) + \frac{8}{3} \alpha, \]  

\[ C = C_0 + (\rho d_1 + c_f d_1^f)II + c_f d_2^f J + c_f d_3^f (Inn + nnI) + c_f d_4^f (J \cdot nn + nn \cdot J + d_5(J \cdot \alpha + \alpha \cdot J), \]

\[ \alpha = \frac{3}{8} \left[ k_0 K^{-1} - \left( 1 + \frac{2}{3} \frac{c_f}{k_0 + k_f} \right) I - \frac{2}{3} c_f \frac{(k_0 - k_f)^2}{k_f} nn \right]. \]

This result may be of interest of its own, since it recovers the information on the crack field (“damage”) from the conductivity data. An important observation is that tensors $\alpha$ and $K$, as well as tensors $\alpha$ and $C$, are, generally, not coaxial in this case. This is due to the fact that the overall anisotropy is caused not only by cracks but by fibers as well.

Introducing $\alpha$ into (6.13) will yield, in a straightforward way, the cross-property correlation in closed form. Being somewhat lengthy, it is not given here. Note that this correlation gives the effective stiffness in terms of $K$ plus terms that involve $nn$ and $nnnn$. Since the direction $n$ is, generally, not one of the principal directions of $K$, the effective stiffness tensor, generally, possesses no elements of symmetry (elastic anisotropy of the general type).

The cross-property correlation simplifies substantially, with $C$ becoming orthotropic and coaxial to $K$, if the direction $n$ is one of the principal axes of the crack density tensor $\alpha$. In this case, the cross-property correlation has the following form:

\[ C_{1111} - C_{1111}^0 = C_f(d_1^f + d_2^f) + q_f[d_1^c(2 + k_0/k_f) + d_3^c + d_4^c] \]

\[ + \frac{3}{8}(d_1^c + d_3^c + d_4^c)(k_{11} - k_0)/k_{11} \]

\[ + \frac{3}{8}d_1^c(k_{22} - k_0)/k_{22} + \frac{3}{8}d_1^c(k_{33} - k_0)/k_{33}, \]

\[ C_{2222} - C_{2222}^0 = C_f(d_1^f + d_2^f) + q_f[d_1^c(2 + k_0/k_f) + d_3^c + d_4^c] \]

\[ + \frac{3}{8}(d_1^c + d_3^c + d_4^c)(k_{11} - k_0)/k_{11} \]

\[ + \frac{3}{8}(d_1^c + d_3^c + d_4^c)(k_{22} - k_0)/k_{22} \]

\[ + \frac{3}{8}d_1^c(k_{33} - k_0)/k_{33}, \]
\[ C_{3333} - C_{3333}^0 = C_f (d_1^f + d_2^f + d_3^f + d_4^f) \]
\[ + q_f [d_1^c (2 + k_0/k_f) + (d_2^c + d_3^c) k_0/k_f] \]
\[ + \frac{3}{8} d_1^c (k_{11} - k_0)/k_{11} + \frac{3}{8} d_2^c (k_{22} - k_0)/k_{22} \]
\[ + \frac{3}{8} (d_1^c + d_3^c + d_4^c) (k_{33} - k_0)/k_{33}, \]
\[ C_{1122} - C_{1122}^0 = C_f [d_1^f (2 + k_0/k_f) + 2 d_3^f] \]
\[ + \frac{3}{8} (d_1^c + d_3^c) (k_{11} - k_0)/k_{11} + \frac{3}{8} (d_2^c + d_4^c) (k_{22} - k_0)/k_{22} \]
\[ + \frac{3}{8} d_1^c (k_{33} - k_0)/k_{33}, \]
\[ C_{1133} - C_{1133}^0 = C_f (d_1^f + d_3^f) + q_f [d_1^c (2 + k_0/k_f) + d_3^c (1 + k_0/k_f)] \]
\[ + \frac{3}{8} (d_1^c + d_3^c) (k_{11} - k_0)/k_{11} + \frac{3}{8} d_1^c (k_{22} - k_0)/k_{22} \]
\[ + \frac{3}{8} (d_1^c + d_3^c) (k_{33} - k_0)/k_{33}, \]
\[ C_{2233} - C_{2233}^0 = C_f (d_1^f + d_3^f) + q_f [d_2^c (2 + k_0/k_f) + d_4^c (1 + k_0/k_f)] \]
\[ + \frac{3}{8} d_1^c (k_{11} - k_0)/k_{11} + \frac{3}{8} d_1^c (k_{22} - k_0)/k_{22} \]
\[ + \frac{3}{8} d_4^c (k_{33} - k_0)/k_{33}, \]
\[ C_{1212} - C_{1212}^0 = C_f d_2^f + 2 q_f d_4^c + \frac{3}{8} d_4^c (k_{11} - k_0)/k_{11} + \frac{3}{8} d_4^c (k_{22} - k_0)/k_{22}, \]
\[ C_{1313} - C_{1313}^0 = C_f (d_2^f + d_4^f) + q_f d_4^c (1 + k_0/k_f) \]
\[ + \frac{3}{8} d_4^c (k_{11} - k_0)/k_{11} + \frac{3}{8} d_4^c (k_{33} - k_0)/k_{33}, \]
\[ C_{2323} - C_{2323}^0 = C_f (d_2^f + d_4^f) + q_f d_4^c (1 + k_0/k_f) \]
\[ + \frac{3}{8} d_4^c (k_{22} - k_0)/k_{22} + \frac{3}{8} d_4^c (k_{33} - k_0)/k_{33}. \]  

This case covers several practically important situations, as follows.

(A) Cracks are normal to the fibers.

(B) Normals to cracks lie in the plane normal to the fibers (if these normals are randomly oriented in the mentioned plane, \( C \) is transversely isotropic, otherwise, if they have a preferential orientation in this plane, \( C \) is orthotropic).

(C) Cracks are randomly oriented.

7. On the accuracy of the derived cross-property correlations

The cross-property correlations (5.7) hinge on the possibility to represent, with sufficient accuracy, one of the two tensors—either the compliance contribution tensor \( H \) of an inclusion (defined by (2.1)) or the stiffness contribution tensor \( N \) (defined by
(2.2)) in terms of a certain symmetric second rank tensor $\Omega$, formulas (3.2). The same accuracy will then hold for a similar representation of the effective elastic properties of a solid with many inclusions (in terms of second rank tensor $\alpha$, formulas (5.4)), provided the following two conditions are satisfied:

- The inclusion shapes are not correlated with either orientations of the inclusions or their volumes (note that the distribution over volumes and over orientations may be correlated).
- The effective properties are considered in the non-interaction approximation.

As far as the first condition is concerned, the requirement of the absence of correlations may be substantially relaxed. For porous materials, as shown by Kachanov et al. (2001), the cross-property correlation can be extended, in many cases, to microstructures comprising two or three families of pores with distinctly different pore statistics. This result can be directly extended to the matrix composites.

We now examine the accuracy of the established cross-property correlations, i.e. estimate the maximal errors caused by the approximate representation of fourth rank tensors $H$ and $N$ in terms of a second rank tensor. The main three factors affecting the accuracy are:

(A) Elastic contrast between the matrix and the inclusion—the ratio $E_0/E_\ast$.
(B) Inclusion shape (aspect ratio for a spheroid).
(C) Poisson’s ratios $\nu_0$ and $\nu_\ast$ of the matrix and of the inclusions.

The results on accuracy of the cross-property correlations that cover various combinations of these factors can be conveniently given in the form of “accuracy maps”. They identify various combinations of parameters corresponding to errors that do not exceed a certain threshold (10%, in Figs. 4 and 5).

Fig. 4 is the accuracy map for the cross-property correlation (5.7a) derived by using the compliance contribution tensors of inclusions. It gives the accuracy in terms of the inclusion shapes and of the elastic properties of inclusions: the region bounded by the curves corresponds to the accuracy better than 10%. The errors are estimated by the norm $\max_{ijkl} \| H_{ijkl} - \hat{H}_{ijkl} \| / \| H_{ijkl} \|$ introduced in Section 3 (its smallness implies that strain responses to all stress states of the actual and of the fictitious inclusions are close). Fig. 5 is a similar accuracy map for the cross-property correlation (5.7b) derived by using the stiffness contribution tensors of inclusions (the errors are estimated by the similar norm $\max_{ijkl} \| N_{ijkl} - \hat{N}_{ijkl} \| / \| N_{ijkl} \|$).

Comparison of Figs. 4 and 5 shows that correlations (5.7b) obtained by using the stiffness contribution tensors of inclusions are, generally, more accurate than the ones obtained by using the compliance contribution tensors (although they are given by somewhat lengthier formulas). It is seen that the cross-property correlations obtained in the framework of either of these dual formulations are more accurate in the case of inclusions that are softer than the matrix (as compared to the case of stiffer ones).

In the case of pores, the accuracy maps are given in Fig. 6. It is seen that the accuracy is quite good in this case: the correlation obtained by using the stiffness contribution tensors $N$ of pores has accuracy better than 4% in all cases.
Fig. 4. Accuracy maps for the approximate representation of the compliance contribution tensor $H$ of an inclusion in terms of a second rank tensor (3.1a). The combinations of parameters (elastic contrast $E_\ast/E_0$ and aspect ratio $\gamma$) corresponding to accuracy better than 10% lie in regions centered at point 1 and bounded by the curves shown.

8. Conclusions

The derived cross-property correlations interrelate, in the closed form, the effective conductivities and the elastic constants of anisotropic composites. They may be valuable for applications, if one property (say, conductivities) is easier to measure than the other one (say, a full set of anisotropic elastic constants). Such correlations may also be used to optimize the microstructure for the best combined conductive/elastic performance.

The derived correlations are approximate. Their accuracy depends on the inclusion shapes and on the elastic contrast between the phases; it remains good over a relatively wide range of parameters (“accuracy maps” of Section 7). In this range, the effective elastic properties are approximately orthotropic (for any orientational distribution of inclusions) and the orthotropy axes are coaxial to the principal axes of conductivity.
Fig. 5. Accuracy maps for the approximate representation of the stiffness contribution tensor $N$ of an inclusion in terms of a second rank tensor (3.1b). The combinations of parameters (elastic contrast $E_*/E_0$ and aspect ratio $\gamma$) corresponding to accuracy better than 10% lie in regions centered at point 1 and bounded by the curves shown.

The correlations are particularly accurate (almost exact) for two-phase composites with one phase constituted by pores/microcracks. In this case, the error is less than 4% in the entire range of pore shapes, and is typically even smaller.

For three-phase composites, the approximate orthotropy of the effective elastic properties and coaxiality of the elasticity and conductivity tensors, generally, do not hold. The cross-property correlations for such composites are examined in detail in the case when one of the phases is constituted by microcracks.

Our results are given in closed form that explicitly reflects inclusion shapes. They are derived in the non-interaction approximation. However, they can be reformulated, in a straightforward way, in the framework of the commonly used approximate schemes (self-consistent, differential, Mori-Tanaka’s) that place non-interacting inclusions into some sort of “effective environment” (effective matrix or effective field).
Fig. 6. The case of pores. Accuracy of the approximate representation of the pore compliance tensor $H$ (a) and pore stiffness tensor $N$ (b) as a function of pore aspect ratio $\gamma$ for several values of $\nu_0$. Note a much higher accuracy (better than 4% in all cases) for tensor $N$.

The derived correlations contain factors that depend on the average inclusion shapes. Their presence reflects the fact that inclusion shapes affect the elasticity and the conductivity differently; otherwise, the correlations would have been universal, independent of microgeometries. However, the information on the microstructure that is reflected in these factors is much less detailed than that required for a direct expression of the effective properties in terms of the microstructure (for example, knowledge of the orientational distribution of inclusions is not needed). The practical utility of the derived cross-property correlations lies precisely in this fact: if the conductivities (or elastic constants) have been measured, then the microstructural information needed to estimate the elastic constants (or conductivities) is rather minimal and approximate.

**Acknowledgements**

This work was partially supported by DOE through a grant to Tufts University.
Appendix A. Tensorial basis in the space of transversely isotropic fourth rank tensors: representation of certain transversely isotropic tensors in terms of the tensorial basis.

The operations of analytic inversion and multiplication of fourth rank tensors are conveniently done in terms of special tensorial bases that are formed by combinations of unit tensor $\delta_{ij}$ and one or two orthogonal unit vectors (see Kunin, 1983; Kanaun and Levin, 1993). In the case of the transversely isotropic elastic symmetry, the following basis is most convenient (it differs slightly from the one used by Kanaun and Levin, 1993):

\[
T^{(1)}_{ijkl} = \theta_{ij}\theta_{kl}, \quad T^{(2)}_{ijkl} = (\theta_{ik}\theta_{lj} + \theta_{il}\theta_{kj} - \theta_{ij}\theta_{kl})/2, \quad T^{(3)}_{ijkl} = \theta_{ij}m_km_l,
\]

\[
T^{(4)}_{ijkl} = m_im_j\theta_{kl}, \quad T^{(5)}_{ijkl} = (\theta_{ik}m_jm_l + \theta_{il}m_km_j + \theta_{jk}m_lm_i + \theta_{jl}m_lm_i)/4,
\]

\[
T^{(6)}_{ijkl} = m_im_jm_km_l,
\]

(A.1)

where $\theta_{ij} = \delta_{ij} - m_im_j$ and $m = m_1e_1 + m_2e_2 + m_3e_3$ is a unit vector along the axis of transverse symmetry.

These tensors form a closed algebra with respect to the operation of (non-commutative) multiplication (contraction over two indices):

\[
(T^{(z)}: T^{(\beta)})_{ijkl} \equiv T^{(z)}_{ijpq} T^{(\beta)}_{pqkl}
\]

(A.2)

The table of multiplication of these tensors has the following form (the column represents the left multipliers):

<table>
<thead>
<tr>
<th>$T^{(1)}$</th>
<th>$T^{(2)}$</th>
<th>$T^{(3)}$</th>
<th>$T^{(4)}$</th>
<th>$T^{(5)}$</th>
<th>$T^{(6)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T^{(1)}$</td>
<td>2 $T^{(1)}$</td>
<td>0</td>
<td>2 $T^{(3)}$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$T^{(2)}$</td>
<td>0</td>
<td>$T^{(2)}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$T^{(3)}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$T^{(1)}$</td>
<td>0</td>
</tr>
<tr>
<td>$T^{(4)}$</td>
<td>2 $T^{(4)}$</td>
<td>0</td>
<td>2 $T^{(6)}$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$T^{(5)}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$T^{(5)}/2$</td>
</tr>
<tr>
<td>$T^{(6)}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$T^{(4)}$</td>
<td>0</td>
</tr>
</tbody>
</table>

Then the inverse of any fourth rank tensor $X$, as well as the product $X : Y$ of two such tensors are readily found in the closed form, as soon as the representations in the basis

\[
X = \sum_{k=1}^{6} X_k T^{(k)}, \quad Y = \sum_{k=1}^{6} Y_k T^{(k)}
\]

(A.3)

are established. Indeed:

(a) Inverse tensor $X^{-1}$ defined by $X^{-1}_{ijmn} X_{mnkl} = (X^{-1}_{ijmn} X^{-1}_{mnkl}) = J_{ijkl}$ is given by

\[
X^{-1} = \frac{X_6}{2A} T^{(1)} + \frac{1}{X_2} T^{(2)} - \frac{X_3}{A} T^{(3)} - \frac{X_4}{A} T^{(4)} + \frac{4}{X_5} T^{(5)} + \frac{2X_1}{A} T^{(6)},
\]

(A.4)

where $A = 2(X_1X_6 - X_3X_4)$. 
(b) product of two tensors $X : Y$ (tensor with $ijkl$ components equal to $X_{ijmn}Y_{mnkl}$) is

$$X : Y = (2X_1 Y_1 + X_3 Y_4) T^{(1)} + X_2 Y_2 T^{(2)} + (2X_1 Y_3 + X_3 Y_6) T^{(3)}$$

$$+ (2X_4 Y_1 + X_6 Y_4) T^{(4)} + \frac{1}{2} X_5 Y_5 T^{(5)} + (X_6 Y_6 + 2X_4 Y_3) T^{(6)}. \quad (A.5)$$

If $x_3$ is the axis of transverse symmetry, tensors $T^{(1)}, \ldots, T^{(6)}$ given by (A.1) have the following non-zero components:

$$T^{(1)}_{1111} = T^{(1)}_{2222} = T^{(1)}_{1122} = T^{(1)}_{2211} = 1,$$

$$T^{(2)}_{1212} = T^{(2)}_{2121} = T^{(2)}_{1221} = T^{(2)}_{2112} = T^{(2)}_{1111} = T^{(2)}_{2222} = \frac{1}{2},$$

$$T^{(2)}_{1122} = T^{(2)}_{2211} = -\frac{1}{2},$$

$$T^{(3)}_{1133} = T^{(3)}_{2233} = 1,$$

$$T^{(4)}_{3311} = T^{(4)}_{3322} = 1,$$

$$T^{(5)}_{1313} = T^{(5)}_{2323} = T^{(5)}_{1331} = T^{(5)}_{2332} = T^{(5)}_{3113} = T^{(5)}_{3131} = T^{(5)}_{3223} = \frac{1}{4},$$

$$T^{(6)}_{3333} = 1. \quad (A.6)$$

A general transversely isotropic fourth rank tensor, being represented in this basis

$$\Psi_{ijkl} = \sum \psi_m T^m_{ijkl}$$

has the following components:

$$\psi_1 = (\Psi_{1111} + \Psi_{1122})/2, \quad \psi_2 = 2\Psi_{1212}, \quad \psi_3 = \Psi_{1133}, \quad \psi_4 = \Psi_{3311},$$

$$\psi_5 = 4\Psi_{1313}, \quad \psi_6 = \Psi_{3333}. \quad (A.7)$$

Utilizing (A.7) one obtains the following representations.

- Tensor of elastic compliances of the isotropic material $S_{ijkl} = \sum s_m T^m_{ijkl}$ has the following components:

$$s_1 = \frac{1 - \nu}{4G(1 + \nu)}, \quad s_2 = \frac{1}{2G}, \quad s_3 = s_4 = -\frac{\nu}{2G(1 + \nu)}, \quad s_5 = \frac{1}{G}, \quad s_6 = \frac{1}{2G(1 + \nu)}. \quad (A.8)$$

- Tensor of elastic stiffness of the isotropic material by $C_{ijkl} = \sum c_m T^m_{ijkl}$ has components

$$c_1 = \lambda + G, \quad c_2 = 2G, \quad c_3 = c_4 = \lambda, \quad c_5 = 4G, \quad c_6 = \lambda + 2G, \quad (A.9)$$

where $\lambda = 2G\nu/(1 - 2\nu)$.

- Unit fourth rank tensors are represented in the form

$$J^{(1)}_{ijkl} = (\delta_{ik}\delta_{lj} + \delta_{ij}\delta_{kl})/2 = \frac{1}{2} T^1_{ijkl} + T^2_{ijkl} + 2T^5_{ijkl} + T^6_{ijkl}, \quad (A.10)$$

$$J^{(2)}_{ijkl} = \delta_{ij}\delta_{kl} = T^1_{ijkl} + T^3_{ijkl} + T^4_{ijkl} + T^6_{ijkl}. \quad (A.11)$$
Eshelby’s tensor for spheroidal inclusion $s_{ijkl} = \sum s^e_m T^m_{ijkl}$ has components

$$
\begin{align*}
  s_1^e &= \frac{1}{2(1-v)} f_0 + f_1, \\
  s_2^e &= \frac{3 - 4v}{2(1-v)} f_0 + f_1, \\
  s_3^e &= \frac{v}{1-v} f_0 - 2f_1, \\
  s_4^e &= \frac{v}{1-v} (1 - 2f_0) - 2f_1, \\
  s_5^e &= 2(1 - f_0 - 4f_1), \\
  s_6^e &= 1 - 2f_0 + 4f_1,
\end{align*}
$$

(A.12)

where $f_0$ and $f_1$ are given by (2.11).

Appendix B.

The coefficients $d_i^c$ and $d_i^f$ entering expressions (6.13) and (6.15) are given by the following formulas:

$$
\begin{align*}
  d_1^c &= \frac{-64G_0(1-v_0)}{3(4 - 9v_0 + 8v_0^2)} \frac{v_0^2 (2 - 5v_0 + 6v_0^2 - 2v_0^3)}{(1 - 2v_0)^2}, \\
  d_3^c &= \frac{-64G_0(1-v_0)}{3(4 - 9v_0 + 8v_0^2)} \frac{2v_0 (2 - 4v_0 + 3v_0^2)}{1 - 2v_0}, \\
  d_4^c &= \frac{-64G_0(1-v_0)}{3(4 - 9v_0 + 8v_0^2)} \frac{(1 - 2v_0 + v_0^2 + v_0^3)}{1 - 2v_0}, \\
  d_1^f &= n_1 - n_2/2, \\
  d_2^f &= n_2, \\
  d_3^f &= 2n_3 + n_2 - 2n_1, \\
  d_4^f &= n_5 - 2n_2, \\
  d_5^f &= n_6 + n_1 + n_2/2 - 2n_3 - n_5,
\end{align*}
$$

(B.1)

where $n_i$ are expressed in terms of elastic constants $G_0$ and $\lambda_0$ of the matrix and of the fiber/matrix elastic contrasts $\delta\lambda = \lambda^* - \lambda_0$, $\delta G = G^* - G_0$ as follows:

$$
\begin{align*}
  n_1 &= \frac{(\delta\lambda + \delta G)(\lambda_0 + G_0)}{(\lambda_0 + 2G_0) + (\delta\lambda + \delta G)}, \\
  n_2 &= \frac{4\delta GG_0(\lambda_0 + 2G_0)}{2G_0(\lambda_0 + 2G_0) + \delta G(\lambda_0 + 3G_0)}, \\
  n_3 &= \frac{\delta\lambda(\lambda_0 + 2G_0)}{(\lambda_0 + 2G_0) + (\delta\lambda + \delta G)}, \\
  n_5 &= 8\delta GG_0/(2G_0 + \delta G), \\
  n_6 &= \frac{(\delta\lambda + 2\delta G)(\lambda_0 + G_0) + 2G_1(3\delta\lambda + 2\delta G)}{(\lambda_0 + 2G_0) + (\delta\lambda + \delta G)}.
\end{align*}
$$

(B.3)
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