Low-Dimensional Modeling for Spatially Developing Free Shear Layers

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The aim of this work is to develop nonlinear low-dimensional models to describe vortex dynamics in spatial shear layers under periodic excitations. By allowing a free variable $g(x)$ to dynamically describe downstream thickness spreading, we are able to obtain base functions in a scaled reference frame and construct effective models with only a few modes in the new space. To apply this modified version of proper orthogonal decomposition (POD)/Galerkin projection, we first scale the flow along $x$ direction (downstream) to match a template function. In the scaled space, the first POD mode can capture more than 80% energy for each frequency. However, to construct Galerkin models carrying most basic dynamics, the second POD mode plays a critical role and needs to be included. Finally, a reconstruction equation for the scaling variable $g$ is derived to relate the scaled space and physical space and to incorporate the physical effects from downstream spreading.

I. Introduction

Free shear layers are often studied as model flow for its simplicity. The behavior of temporally developing (TD) or spatially developing (SD) shear layers have been, of course, studied for several decades from a theoretical, numerical, and experimental perspective. A basic understanding has been developed from previous research. However, shear flows are still too complex to directly apply dynamic systems and control theories which have been widely used to analyze and understand many simple mechanical systems. Though low-dimensional models have been proposed and succeeded in some applications, they are mainly phenomenological. Our goal for this paper is to develop low-dimensional models from the direct projection of governing equations.

Based on a combination of POD/Galerkin projection method and symmetry reduction idea introduced from geometric mechanics, Wei and Rowley recently developed low-dimensional models for TD free shear flows. A scaling factor $g(t)$ was introduced to factor out the thickness growth, so that, similar vortex structures at different thickness can be uniformly described. It should be pointed out that the scaling factor was introduced and calculated simultaneously with no self-similarity pre-known. A related technique has been used for traveling solutions and self-similar solutions. A similar approach is applied in this work for SD free shear flows, though many new challenges appear from the difference, sometimes critical, between temporal and spatial development.

For SD flows, since the self-similarity happens streamwise ($x$ direction), a scaling factor $g(x)$ as a function of $x$ is naturally chosen for symmetry reduction. The SD flow here has been forced periodically in time so

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that the problem can be simplified as in the TD flow case. Accordingly, the final low-dimensional models for current SD shear layers are in form of ordinary differential equations evolving in $x$ instead of $t$.

One major difference/difficulty in modeling SD shear layers is that the Navier-Stokes equations are parabolic in time but not parabolic in space. The ellipticity in space essentially makes models marching in space ill-posed! To justify the usage of the current model reduction methodology, parabolization of the original Navier-Stokes equations has to be done first with suitable approximation.

The paper is arranged as follows. Direct numerical simulation of SD shear layers is described in section II. Equation parabolization for modeling will be discussed in section III. Section IV will then describe the entire methodology of low-dimensional modeling of SD shear layers. The results with further comparison and discussions will be in section V.

II. Simulation of spatial-developing shear layers

The flow considered in this paper is a two-dimensional free shear layer developing spatially shown schematically in figure 1. With the characteristic length being the initial vorticity thickness $\delta_\omega = \Delta U / |du/dy|_{\text{max}}$ at the entrance and the characteristic velocity being the maximum velocity change across the shear layer $\Delta U$, Reynolds number of the flow is $Re = 200$. All values are non-dimensionalized by the same characteristic values for the rest of the paper. Though the modeling will be based on the equations derived from incompressible Navier-Stokes equations, the simulation itself solves the fully compressible Navier-Stokes equations at low Mach number (convective Mach number is 0.3 in average), using a code that has been validated before in previous work. The velocity divergence induced by weak compressibility here is small, and therefore has negligible impact on current modeling if no compressibility-specific features (e.g. acoustics) are considered.

![Figure 1. Schematic of the two-dimensional free shear layer simulation.](image)

The flow is simulated in a domain extending 200 in $x$ and out to $\pm 80$ in $y$ from the mixing layer. Extra buffer areas with 20 at the top and bottom and 60 to the left and right are applied in the computation. To make the flow periodic in time, we seed bodyforce excitation in a small box area $5 < x < 15$, $-5 < y < 5$ to trigger the instability. Two excitation frequencies $k = 1$ and $k = 2$ are picked based on non-dimensional time period $T = 38.4$ for later analysis.
III. Parabolization of governing equations

The governing equations for two-dimensional incompressible flow are

\[
\begin{align*}
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0 \\
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} &= -\frac{\partial p}{\partial x} + \frac{1}{Re} \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \\
\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} &= -\frac{\partial p}{\partial y} + \frac{1}{Re} \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right).
\end{align*}
\]

Mathematically, the elliptic behavior in space is introduced in two manners: 1) the Laplacian of velocity in viscous terms; and 2) pressure Poisson equation implied by the pressure terms and the continuity equation. The first one can be removed in a relatively easy way by assuming a thin layer, which ignores all \(\frac{d^2(\cdot)}{dx^2}\) terms. For the second one, with the outer flow \(U(x)\) being a function of \(x\) only, the thin-layer assumption can also remove the pressure effects. The momentum equations become

\[
\begin{align*}
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} &= \frac{1}{Re} \frac{\partial^2 u}{\partial y^2} \\
\frac{\partial p}{\partial y} &= 0.
\end{align*}
\]

However, equation (2) loses the instability features\(^{11, 12}\) required to initiate/terminate basic vortex dynamics. So, we put the terms back in \(y\) momentum equation while still assuming negligible effects from pressure gradient and \(\frac{d^2(\cdot)}{dx^2}\) terms for a thin layer. Thus, the reduced parabolic equations for later modeling are

\[
\begin{align*}
\frac{\partial u}{\partial x} &= -v \frac{\partial u}{\partial y} - \frac{\partial u}{\partial t} + \frac{1}{Re} \frac{\partial^2 u}{\partial y^2} \\
\frac{\partial v}{\partial x} &= -v \frac{\partial v}{\partial y} - \frac{\partial v}{\partial t} + \frac{1}{Re} \frac{\partial^2 v}{\partial y^2},
\end{align*}
\]

where terms are rearranged for convenience. It is understood that pressure can be viewed as a Lagrange multiplier in the momentum equations to reinforce the continuity equation (mass conservation). The simplification in (3) apparently releases this constraint, that is, the system mass can drift away. In our previous modeling for TD shear flow, we noticed that the model without mass conservation\(^{13}\) is less accurate than the model with mass conservation,\(^6\) however, the mass drift is slow and does not ruin the dynamics entirely and the models based on momentum equations only (without pressure terms) can still keep the fundamental dynamics fairly well. For SD shear flow, we would expect the same. We want to emphasize that though the projection is made upon the reduced equations, the flow simulation still uses original Navier-Stokes equations which assure the accuracy of base functions (modes).

IV. Low-dimensional models

IV.A. Scaling the flow dynamically

A common approach to low-dimensional modeling is to project the governing equations onto a fixed set of basis functions, which are determined mathematically (i.e. Fourier modes) or empirically (i.e. POD modes). Here, since the shear layer thickness is spreading downstream by vortex roll-up, vortex pairing and merging, Reynolds stresses, and viscous dissipation, we consider basis functions that can scale in \(y\)-direction to accommodate the spreading. A similar idea has been successfully applied on temporally-developing periodic shear flows.\(^6, 13\) The main difference of the current scaling is that the scaling function becomes a function of \(x\) instead of \(t\). If the velocity vector is defined by \(q = (u \ v)^T\), a scaled variable \(\tilde{q}\) can be introduced by

\[
q(x, y, t) = \tilde{q}(x, g(x)y, t),
\]

where
where $g(x)$ is a scaling factor to be determined. The purpose of introducing $g(x)$ is to factor out the mean flow development so that the flow dynamics can be represented by much fewer modes. Consequently, $g(x)$ is defined here to line up the scaled solution $\tilde{q}$ the best to a pre-selected template function. The initial shear flow profile $q_0 = (u_0, v_0)$ can be a natural choice for this template, where

$$u_0 = U_1 + \frac{U_2 - U_1}{2}(1 + \tanh(2y)), \quad v_0 = 0. \quad (5)$$

It is noticed that the only non-zero component in the template is $u_0$, the scaling factor $g(x)$ is therefore defined by

$$g(x) = \arg\min_g \|u(x, y, t) - u_0(y)\|^2, \quad (6)$$

where $\| \cdot \|^2$ is an $L^2$ norm defined upon the integration over $y$ and single time period. A new thickness definition $\delta_g$ is introduced by

$$\delta_g(t) = \frac{1}{g(t)}, \quad (7)$$

which can be used to measure the shear layer spreading as an alternative to vortex thickness or momentum thickness. The condition for $\tilde{u}(x, y, t) = u(x, y/g, t)$ to always match the template the best can be written as

$$\frac{d}{ds}\Bigg|_{s=0} \|\tilde{u}(x, y, t) - u_0(h(s)y)\|^2 = 0,$$

where $h(s)$ is any curve in $\mathbb{R}^+$ with $h(0) = 1$, and the same norm on the space of functions of $(y, t)$ is used: that is, $h = 1$ is a local minimum of the error norm above. This expression becomes

$$-2 \left\langle \frac{d}{ds}\Bigg|_{s=0} u_0(h(s)y), \tilde{u}(x, y, t) - u_0(y) \right\rangle = 0,$$

which becomes

$$\left\langle y \frac{\partial u_0}{\partial y}, \tilde{u} - u_0 \right\rangle = 0. \quad (8)$$

Geometrically, this result means that the set of all such functions $\tilde{u}$ that are scaled so that they most closely match the template $u_0$ is an affine space through $u_0$ and orthogonal to $y\partial_y u_0$.

**IV.B. Equations in scaled space**

We regard the parabolized equations as a dynamical system evolving on a function space $H$, consisting of the flow variables at $(y, t)$ marching in $x$. Thus, $q(x) \in H$ is a snapshot of the entire flow at location $x$, and equations (3) may be written as

$$\mathbf{A} \frac{\partial q(x)}{\partial x} = f(q(x)), \quad (9)$$

where

$$\mathbf{A} = \begin{bmatrix} u & 0 \\ 0 & u \end{bmatrix}, \quad (10)$$

to form the left-hand-side of (3), and $f$ is a differential operator on $H$ such that $f(q(x))$ gives all right-hand-side terms of (3). Here matrix $\mathbf{A}$ can also be written in scaled variables as

$$\tilde{\mathbf{A}} = \begin{bmatrix} \tilde{u} & 0 \\ 0 & \tilde{u} \end{bmatrix}, \quad (11)$$

and the same scaling holds as

$$\mathbf{A}(x, y, t) = \tilde{\mathbf{A}}(x, g(x)y, t). \quad (12)$$
If we introduce the scaling operator $S_g : H \rightarrow H$, defined by

$$S_g[q(y, t)] = q(gy, t), \quad \forall g \in \mathbb{R}^+$$

(13)

then the scaling (4) becomes $q = S_g[q]$, (12) becomes $A = S_g[\hat{A}]$, and the governing equations (9) may be written

$$S_{g(x)}[\hat{A}(x, y, t)] \frac{\partial}{\partial x} S_{g(x)}[\hat{q}(x, y, t)] = f(S_g[q(x, y, t)]).$$

(14)

Since

$$\frac{\partial}{\partial x} S_{g(x)}[\hat{q}(x, y, t)] = \frac{\partial}{\partial x} \hat{q}(x, gy, t) = \frac{\partial \hat{q}}{\partial x} + \dot{g} y \frac{\partial \hat{q}}{\partial y},$$

(15)

(14) becomes

$$S_g[\hat{A}] \frac{\partial \hat{q}}{\partial x} = f(S_g[q]) - \dot{g} y \frac{\partial \hat{q}}{\partial y}.$$  

(16)

If we define $S_{1/g}$ an inverse mapping of $S_g$ and $f_g(\hat{q}) = S_{1/g}f(S_g[q])$, applying the inverse mapping to above equation, we have the governing equations in scaled space

$$\hat{A} \frac{\partial \hat{q}}{\partial x} = f_g(\hat{q}) - \hat{A} \frac{\dot{g} y}{g} \frac{\partial \hat{q}}{\partial y}.$$  

(17)

Equation (17) can be written separately in variables $\hat{u}$ and $\hat{v}$ as

$$\frac{\partial \hat{u}}{\partial x} = f^1_g \frac{\dot{g} y}{g} \frac{\partial \hat{u}}{\partial y},$$

$$\frac{\partial \hat{v}}{\partial x} = f^2_g \frac{\dot{g} y}{g} \frac{\partial \hat{v}}{\partial y}.$$  

(18)

However these equations alone are not sufficient to evolve the dynamics without the knowledge of $g(x)$.

**IV.C. Equation for scaling variable**

In the section, the evolution equation for $g(x)$ will be derived to close the system. Differentiating the constraint (8) along $x$, we have

$$\left\langle y \frac{\partial u_0}{\partial y}, \frac{\partial \hat{u}}{\partial x} \right\rangle = 0.$$  

(19)

To avoid complex nonlinear terms introduced by $1/\hat{u}$ in projection, we approximate the derivative of $\hat{u}$ here from the $x$-momentum equation in (18) as:

$$\frac{\partial \hat{u}}{\partial x} \approx \frac{f^1_g}{u_0} \frac{\dot{g} y}{g} \frac{\partial \hat{u}}{\partial y}.$$  

(20)

So that, we have

$$\left\langle y \frac{\partial u_0}{\partial y}, \frac{f^1_g}{u_0} \frac{\dot{g} y}{g} \frac{\partial \hat{u}}{\partial y} \right\rangle = 0,$$

(21)

which becomes

$$\frac{\dot{g}}{g} = \left\langle \frac{f^1_g}{u_0} \frac{\partial u_0}{\partial y}, \frac{\partial \hat{u}}{\partial y} \right\rangle.$$  

(22)

Altogether, equation (17) for $\hat{q}$ and equation (22) for $g$ define the system evolution in the scaled frame.
IV.D. Galerkin projection

Before we implement the projection, it is necessary to group the equation right-hand-side into nonlinear and linear terms as

$$f(q) = N(q,q) + L(q),$$

where

$$N(q,q) = \begin{bmatrix} -u \frac{\partial u}{\partial y} \\ -v \frac{\partial v}{\partial y} \end{bmatrix}, \quad L(q) = \begin{bmatrix} -\frac{\partial \hat{u}}{\partial t} + \frac{1}{Re} \frac{\partial^2 \hat{u}}{\partial y^2} \\ -\frac{\partial \hat{v}}{\partial t} + \frac{1}{Re} \frac{\partial^2 \hat{v}}{\partial y^2} \end{bmatrix}.$$  

The corresponding terms in scaled frame are

$$f_g(\tilde{q}) = N_g(\tilde{q},\tilde{q}) + L_g(\tilde{q}),$$

where

$$N_g(\tilde{q},\tilde{q}) = \begin{bmatrix} -\tilde{u} \frac{\partial \tilde{u}}{\partial y} \\ -\tilde{v} \frac{\partial \tilde{v}}{\partial y} \end{bmatrix}, \quad L_g(\tilde{q}) = \begin{bmatrix} -\frac{\partial \tilde{u}}{\partial t} + \frac{1}{Re} \frac{\partial^2 \tilde{u}}{\partial y^2} \\ -\frac{\partial \tilde{v}}{\partial t} + \frac{1}{Re} \frac{\partial^2 \tilde{v}}{\partial y^2} \end{bmatrix}.$$  

We can then expand \( \tilde{q} \) in its base functions as

$$\tilde{q} = q_0(y) + \sum_{k=-\infty}^{+\infty} \sum_{n=0}^{\infty} a_{k,n}(x) \Phi_{k,n}(t,y).$$

where

$$\Phi_{k,n}(t,y) = e^{2\pi ikt/T} \phi_{k,n}(y).$$

Here, \( k \) is the frequency, \( T \) is the time period, and \( \phi_{k,n} = (\hat{u}_{k,n}, \hat{v}_{k,n}) \) is the \( n \)th POD mode for frequency \( k \).

The energy of each POD mode \((k,n)\) is quantified by

$$\lambda_{k,n} = \frac{1}{2} |\langle q - q_0, \Phi_{k,n} \rangle|^2 = a_{k,n}^2,$$

where \( \cdot \) denotes a streamwise spatial average. We start with simple case retaining only frequencies \( k = \pm 1 \), and the first two POD modes \( n = 1 \) and \( n = 2 \) for each frequency. The summation is then an approximation of the original \( \tilde{q} \). We will retain the notation \( \tilde{q} \) for the finite sum in (27). Since \( \tilde{q} \) must be real, we have the additional constraint that

$$a_{1,1} \Phi_{1,1} + a_{1,2} \Phi_{1,2} = a_{-1,1}^* \Phi_{-1,1} + a_{-1,2}^* \Phi_{-1,2}$$

which permits further simplification of the equations that follow.

To obtain the equations for coefficients \( a_{1,1}(x) \) and \( a_{1,2}(x) \), we project the governing equation (17) onto modes \( \Phi_{1,1} \) and \( \Phi_{1,2} \). Eventually, the spatial evolution equation for coefficient vector \( a = (a_{1,1} \ a_{1,2})^T \) is

$$B \dot{a} = (gC + \Lambda + \frac{1}{Re} g^2 D + \frac{\dot{g}}{g} E) a.$$  

Matrices \( B, C, \Lambda, D, \) and \( E \) are defined by

$$B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}, \quad C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}, \quad \Lambda = \begin{bmatrix} c_{13} & 0 \\ 0 & c_{23} \end{bmatrix},$$

$$D = \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix}, \quad E = \begin{bmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{bmatrix},$$

where all coefficients in matrices are well-defined and listed in appendix A with some detailed derivation.

To close the system, the spatial evolution for thickness adjustment \( g(x) \) is needed and can be obtained from scaling relation (22) with the same two modes being retained:

$$\dot{g} = (c_{01} a_{1,1}^* a_{1,1}^* + c_{02} a_{1,1} a_{1,2}^* + c_{03} a_{1,2} g_{1,1} + c_{04} a_{1,2}^* a_{1,2}^*) \frac{1}{b_0} g^2 + \frac{1}{Re b_0} g^3,$$

where all coefficients are also defined in appendix A. Having (33) together with (31), we are able to solve this low-dimensional model system.
V. Results and discussions

With the numerical configuration mentioned in section II, we start the simulation with excitations on hyperbolic tangent velocity profile and then allow a transition time for as long as 10 time periods of $k = 1$ frequency in our case. Finally, another 10 periods are simulated and provide the data for mode decomposition.

Figure 2 shows several snapshots taken from the simulation, where typical dynamics such as vortex roll-up, pairing, and merging can be clearly observed. To have a more quantitative picture of the thickness growth, we compute $g$ thickness as the flow develops downstream (figure 3). Similar to our observation in TD shear layers, the thickness has overall growth with viscous spreading while showing events associated with vortices.

The simulation data were then mapped to a scaled space with downstream thickness growth being factored out. In the scaled space, we can easily get POD modes for each time-frequency. Figure 4 shows the first and second POD modes of frequency $k = 1$. Their shape looks very similar to the POD modes of space-frequency $k = 1$ in previous TD cases.

Table 1 lists each modes’ energy defined by (29). Similar to TD cases, the first POD modes of each frequency contain most of the energy (totally 81.0% by mode (1,1) and (2,1)). However, as we have learned clearly from TD modeling, the second POD modes are also dynamically important despite their much less energy (totally 8.1% by mode (1,2) and (2,2)). For current 2-mode model, we use mode (1,1) and (1,2) which contain about 72% of the total energy.

<table>
<thead>
<tr>
<th>$(k, n)$</th>
<th>energy (%)</th>
<th>$(k, n)$</th>
<th>energy (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1,1)</td>
<td>66.3</td>
<td>(2,1)</td>
<td>14.7</td>
</tr>
<tr>
<td>(1,2)</td>
<td>5.4</td>
<td>(2,2)</td>
<td>2.7</td>
</tr>
</tbody>
</table>

Table 1. Energy contained by different modes.

Figure 5 shows the projection of DNS data onto these 4 modes.

Finally, to get a 2-mode model of SD shear layer, we substitute the first and second POD modes of $k = 1$ into the coefficients and models defined in section IV and Appendix A, and evolve the model equations along $x$ to get the coefficients $a$ and scaling function $g$ as the flow develops downstream. Figure 6 compares the
Figure 3. The thickness growth along $x$ direction while the shear layer is developing: sample vortex structure is shown for comparison.

Figure 4. The absolute value $|\hat{u}|$ and $|\hat{v}|$ for POD modes. Left: $(k, n) = (1, 1)$. Right: $(k, n) = (1, 2)$. 
mode coefficients and $g$ thickness calculated from the model to the DNS results. The model captures some basic dynamics: 1) overall thickness growth; 2) oscillation frequency and amplitude of each modes. However, the model certainly over-estimated the rapid increase at the beginning when $k = 1$ first appears. This is not surprising. Similar over-estimation has been noticed in 2-mode model of TD shear layer when mass conservation was not reenforced. This error has been much reduced in 4-mode model\cite{13} and in 2-mode model with mass conservation applied.\cite{6} We would expect the same improvement for SD shear layers in future work. A more complex 4-mode model is under development, and is expected to describe more complex dynamics involving both $k = 1$ and $k = 2$ frequencies.
VI. Conclusion

Direct numerical simulation of a SD shear layer shows overall downstream spreading with events marked by vortex roll-up, pairing, and merging. At most of time, when vortex structures translate downstream, most physics keeps the same. However, this similarity can be damaged by the effects from mean flow variation along $x$, and therefore, the number of required modes increases in reduced-order modeling. In order to avoid this problem, we introduce a function $g(x)$ to scale the flow dynamically in $y$-direction so that the shear-layer thickness remains the same in the scaled space. Then, a low-dimensional system evolving downstream can be built with great efficiency by traditional POD/Galerkin projection in the new space. Finally, a reconstruction equation for $g(x)$ is derived and computed simultaneously to close the system.

The approach needs to base on equations parabolized along $x$ and at the same time with enough terms to keep some key physics (i.e. instability). The parabolization is basically handled by thin-layer assumption, while some terms have been put back to keep the instability right.

Eventually, a 2-mode model for SD shear layers is developed to describe vortex roll-up and shear-layer thickness changes. Similar to TD cases, we need at least two POD modes for each frequency for a successful model. The shapes of each modes are also similar to those of TD cases, and therefore, a possible connection between the POD modes and the instability modes is indicated. The downstream development of each modes is depicted fairly well by the 2-mode model without considering mass conservation, though the thickness growth is significantly overestimated. The reason of this overestimation may be the mass drift caused by the omission of pressure terms for parabolization purpose. A 4-mode model (currently in development) or a mass-conserved model (2 or 4-mode) should fix this problem if a situation similar to TD shear layers can happen.

Acknowledgment

We thank Professor Clancy Rowley for helpful discussion. MW and BQ also gratefully acknowledge the support from Sandia-University Research Program (SURP). Sandia is a multiprogram laboratory operated by Sandia Corporation, a Lockheed Martin Company for the United States Department of Energy’s National Nuclear Security Administration under contract DE-AC04-94AL85000.

A. Appendix

A.A. Projection of left-hand-side

\[
\left\langle \hat{A} \frac{\partial \tilde{q}}{\partial x}, \Phi_{1,1} \right\rangle = T \left[ \int (u_0 \tilde{u}_{1,1}^* \tilde{v}_{1,1}^* + u_0 \tilde{v}_{1,1}^* \tilde{v}_{1,1}^*) dy \right] \tilde{a}_{1,1} + T \left[ \int (u_0 \tilde{u}_{1,2} \tilde{v}_{1,1}^* + u_0 \tilde{v}_{1,2} \tilde{v}_{1,1}^*) dy \right] \tilde{a}_{1,2} \\
= T (b_{11} \tilde{a}_{1,1} + b_{12} \tilde{a}_{1,2}),
\]

so that coefficients are defined

\[
b_{11} = \int (u_0 \tilde{u}_{1,1}^* \tilde{v}_{1,1}^* + u_0 \tilde{v}_{1,1}^* \tilde{v}_{1,1}^*) dy \\
b_{12} = \int (u_0 \tilde{u}_{1,2} \tilde{v}_{1,1}^* + u_0 \tilde{v}_{1,2} \tilde{v}_{1,1}^*) dy.
\]

Similarly,

\[
b_{21} = \int (u_0 \tilde{u}_{1,1}^* \tilde{v}_{1,2}^* + u_0 \tilde{v}_{1,1}^* \tilde{v}_{1,2}^*) dy \\
b_{22} = \int (u_0 \tilde{u}_{1,2} \tilde{v}_{1,2}^* + u_0 \tilde{v}_{1,2} \tilde{v}_{1,2}^*) dy.
\]
A.B. Projection of nonlinear terms

\[
\langle N_g(\hat{q}, \tilde{q}), \Phi_{1,1} \rangle = -T g \left[ \int (\hat{v}_0 \frac{d\tilde{u}_{1,1}}{dy} \hat{u}_{1,1}^* + \hat{v}_0 \frac{d\hat{v}_{1,1}}{dy} \hat{v}_{1,1}^*)dy \right] a_{1,1} \\
- T g \left[ \int (\hat{v}_0 \frac{d\tilde{u}_{1,1}}{dy} \hat{u}_{1,1}^* + \hat{v}_0 \frac{d\hat{v}_{1,1}}{dy} \hat{v}_{1,1}^*)dy \right] a_{1,2} \\
- T g \left[ \int (\hat{v}_{1,1} \frac{d\tilde{u}_{1,1}}{dy} \hat{u}_{1,1}^* + \hat{v}_{1,1} \frac{d\hat{v}_{1,1}}{dy} \hat{v}_{1,1}^*)dy \right] a_{1,1} \\
- T g \left[ \int (\hat{v}_{1,2} \frac{d\tilde{u}_{1,1}}{dy} \hat{u}_{1,1}^* + \hat{v}_{1,2} \frac{d\hat{v}_{1,1}}{dy} \hat{v}_{1,1}^*)dy \right] a_{1,2} \\
= -T g \left[ \int (\hat{v}_{1,1} \frac{d\tilde{u}_{1,1}}{dy} \hat{u}_{1,1}^*)dy \right] a_{1,1} - T g \left[ \int (\hat{v}_{1,2} \frac{d\tilde{u}_{1,1}}{dy} \hat{u}_{1,1}^*)dy \right] a_{1,2},
\]

where \( \hat{v}_0 = 0 \) is applied. So, we have

\[
c_{11} = - \int (\hat{v}_{1,1} \frac{d\tilde{u}_{1,1}}{dy})dy \\
c_{12} = - \int (\hat{v}_{1,2} \frac{d\tilde{u}_{1,1}}{dy})dy.
\]

Similarly,

\[
c_{21} = - \int (\hat{v}_{1,1} \frac{d\tilde{u}_{1,2}}{dy})dy \\
c_{22} = - \int (\hat{v}_{1,2} \frac{d\tilde{u}_{1,2}}{dy})dy.
\]

A.C. Projection of linear terms

\[
\langle L_g(\tilde{q}), \Phi_{1,1} \rangle = -T \left( \frac{2\pi i}{T} \right) \left[ \int (\tilde{u}_{1,1} \hat{u}_{1,1}^* + \hat{v}_{1,1} \hat{v}_{1,1}^*)dy \right] a_{1,1} \\
+ \frac{1}{Re} \epsilon^2 T \left\{ \left[ \int (\frac{d^2 \tilde{u}_{1,1}}{dy^2} \hat{u}_{1,1}^* + \frac{d^2 \hat{v}_{1,1}}{dy^2} \hat{v}_{1,1}^*)dy \right] a_{1,1} \right\} \\
+ \left[ \int (\frac{d^2 \tilde{u}_{1,2}}{dy^2} \hat{u}_{1,1}^* + \frac{d^2 \hat{v}_{1,2}}{dy^2} \hat{v}_{1,1}^*)dy \right] a_{1,2} \right) ,
\]

that

\[
c_{13} = - \left( \frac{2\pi i}{T} \right) \int (\tilde{u}_{1,1} \hat{u}_{1,1}^* + \hat{v}_{1,1} \hat{v}_{1,1}^*)dy \\
d_{11} = \int (\frac{d^2 \tilde{u}_{1,1}}{dy^2} \hat{u}_{1,1}^* + \frac{d^2 \hat{v}_{1,1}}{dy^2} \hat{v}_{1,1}^*)dy \\
d_{12} = \int (\frac{d^2 \tilde{u}_{1,2}}{dy^2} \hat{u}_{1,1}^* + \frac{d^2 \hat{v}_{1,2}}{dy^2} \hat{v}_{1,1}^*)dy.
\]

Similarly

\[
c_{23} = - \left( \frac{2\pi i}{T} \right) \int (\tilde{u}_{1,2} \hat{u}_{1,2}^* + \hat{v}_{1,2} \hat{v}_{1,2}^*)dy \\
d_{21} = \int (\frac{d^2 \tilde{u}_{1,1}}{dy^2} \hat{u}_{1,2}^* + \frac{d^2 \hat{v}_{1,1}}{dy^2} \hat{v}_{1,2}^*)dy \\
d_{22} = \int (\frac{d^2 \tilde{u}_{1,2}}{dy^2} \hat{u}_{1,2}^* + \frac{d^2 \hat{v}_{1,2}}{dy^2} \hat{v}_{1,2}^*)dy.
\]
A.D. Projection of thickness correction terms

The correction term of downstream evolution resulted by the scaling is

\[
\begin{align*}
\left\langle -\frac{A}{g} \frac{\partial y}{dy} \frac{d\tilde{q}}{dy}, \Phi_{1,1} \right\rangle &= -T \frac{\partial y}{dy} \left( \int \left( u_0 \frac{d\tilde{u}_{1,1}}{dy} \tilde{a}^{*}_{1,1} + u_0 \frac{d\tilde{v}_{1,1}}{dy} \hat{v}^{*}_{1,1} \right) dy \right) a_{1,1} \\
&\quad - T \frac{\partial y}{dy} \left( \int \left( u_0 \frac{d\tilde{u}_{1,2}}{dy} \tilde{a}^{*}_{1,2} + u_0 \frac{d\tilde{v}_{1,2}}{dy} \hat{v}^{*}_{1,2} \right) dy \right) a_{1,2} \\
&\quad - T \frac{\partial y}{dy} \left( \int \left( \tilde{u}_{1,1} \frac{d\tilde{u}_{1,1}}{dy} \tilde{a}^{*}_{1,1} + \tilde{u}_{1,1} \frac{d\tilde{v}_{1,1}}{dy} \hat{v}^{*}_{1,1} \right) dy \right) a_{1,1} \\
&\quad - T \frac{\partial y}{dy} \left( \int \left( \tilde{u}_{1,2} \frac{d\tilde{u}_{1,2}}{dy} \tilde{a}^{*}_{1,2} + \tilde{u}_{1,2} \frac{d\tilde{v}_{1,2}}{dy} \hat{v}^{*}_{1,2} \right) dy \right) a_{1,2} \\
&= T \frac{\partial y}{dy} (e_{11} a_{1,1} + e_{12} a_{1,2}) .
\end{align*}
\]

Coefficients are

\[
\begin{align*}
e_{11} &= - \int \left( u_0 \frac{d\tilde{u}_{1,1}}{dy} \tilde{a}^{*}_{1,1} + u_0 \frac{d\tilde{v}_{1,1}}{dy} \hat{v}^{*}_{1,1} + \tilde{u}_{1,1} \frac{d\tilde{u}_{1,1}}{dy} \tilde{a}^{*}_{1,1} \right) dy \\
e_{12} &= - \int \left( u_0 \frac{d\tilde{u}_{1,2}}{dy} \tilde{a}^{*}_{1,2} + u_0 \frac{d\tilde{v}_{1,2}}{dy} \hat{v}^{*}_{1,2} + \tilde{u}_{1,2} \frac{d\tilde{u}_{1,2}}{dy} \tilde{a}^{*}_{1,2} \right) dy .
\end{align*}
\]

where \( \hat{v}_{0} = 0 \) is applied again. Similarly,

\[
\begin{align*}
e_{21} &= - \int \left( u_0 \frac{d\tilde{u}_{1,1}}{dy} \tilde{a}^{*}_{1,2} + u_0 \frac{d\tilde{v}_{1,1}}{dy} \hat{v}^{*}_{1,2} + \tilde{u}_{1,2} \frac{d\tilde{u}_{1,1}}{dy} \tilde{a}^{*}_{1,1} \right) dy \\
e_{22} &= - \int \left( u_0 \frac{d\tilde{u}_{1,2}}{dy} \tilde{a}^{*}_{1,2} + u_0 \frac{d\tilde{v}_{1,2}}{dy} \hat{v}^{*}_{1,2} + \tilde{u}_{1,2} \frac{d\tilde{u}_{1,2}}{dy} \tilde{a}^{*}_{1,2} \right) dy .
\end{align*}
\]

A.E. Terms for thickness evolution

\[
\left\langle \frac{y}{\partial y} \frac{\partial u_0}{\partial y} \right\rangle = T \int \left( \frac{y}{\partial y} \frac{\partial u_0}{\partial y} \right) dy ,
\]

and

\[
\begin{align*}
\left\langle f_4^1, y \frac{\partial u_0}{\partial y} \right\rangle &= \left\langle N_s^1 (\tilde{q}, \hat{q}) , y \frac{\partial u_0}{\partial y} \right\rangle + \left\langle L_s^1 (\tilde{q}), y \frac{\partial u_0}{\partial y} \right\rangle \\
&= - T \frac{\partial y}{dy} \left( \int \left( \tilde{v}_{1,1} \frac{d\tilde{v}_{1,1}}{dy} + \hat{v}_{1,1} \frac{d\hat{v}_{1,1}}{dy} \right) \frac{d\tilde{u}_{1,1}}{dy} \frac{d\tilde{u}_{1,1}}{dy} \right) a_{1,1} a_{1,1} \\
&\quad - T \frac{\partial y}{dy} \left( \int \left( \tilde{v}_{1,2} \frac{d\tilde{v}_{1,2}}{dy} + \hat{v}_{1,2} \frac{d\hat{v}_{1,2}}{dy} \right) \frac{d\tilde{u}_{1,2}}{dy} \frac{d\tilde{u}_{1,2}}{dy} \right) a_{1,1} a_{1,2} \\
&\quad - T \frac{\partial y}{dy} \left( \int \left( \tilde{v}_{1,1} \frac{d\tilde{v}_{1,1}}{dy} + \hat{v}_{1,1} \frac{d\hat{v}_{1,1}}{dy} \right) \frac{d\tilde{u}_{1,1}}{dy} \frac{d\tilde{u}_{1,2}}{dy} \right) a_{1,1} a_{1,1} \\
&\quad - T \frac{\partial y}{dy} \left( \int \left( \tilde{v}_{1,2} \frac{d\tilde{v}_{1,2}}{dy} + \hat{v}_{1,2} \frac{d\hat{v}_{1,2}}{dy} \right) \frac{d\tilde{u}_{1,2}}{dy} \frac{d\tilde{u}_{1,2}}{dy} \right) a_{1,2} a_{1,2} \\
&\quad + \frac{1}{Re} T \int \left( \frac{d^2 u_0}{dy^2} \frac{\partial u_0}{\partial y} \right) dy .
\end{align*}
\]
Therefore, the coefficients in thickness evolution equation (33) are

\[b_0 = \int \left( y \frac{du_0}{dy} \right)^2 dy \]  
\[c_{01} = -\int \left( \hat{v}_{1,1} \frac{d\hat{u}_{1,1}^*}{dy} + \hat{v}_{1,1} \frac{d\hat{u}_{1,1}^*}{dy} \right) y \frac{du_0}{u_0} dy \]  
\[c_{02} = -\int \left( \hat{v}_{1,2} \frac{d\hat{u}_{1,2}^*}{dy} + \hat{v}_{1,2} \frac{d\hat{u}_{1,2}^*}{dy} \right) y \frac{du_0}{u_0} dy \]  
\[c_{03} = -\int \left( \hat{v}_{1,2} \frac{d\hat{u}_{1,1}^*}{dy} + \hat{v}_{1,2} \frac{d\hat{u}_{1,1}^*}{dy} \right) y \frac{du_0}{u_0} dy \]  
\[c_{04} = -\int \left( \hat{v}_{1,2} \frac{d\hat{u}_{1,2}^*}{dy} + \hat{v}_{1,2} \frac{d\hat{u}_{1,2}^*}{dy} \right) y \frac{du_0}{u_0} dy \]  
\[d_0 = \int \left( \frac{d^2 u_0}{dy^2} \right) y \frac{du_0}{u_0} dy \]  

References