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Dynamics and control of the system of a 2-D rigid circular cylinder and point vortices

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***Abstract*— The Hamiltonian system of a 2-D rigid circular cylinder dynamically interacting with N point vortices in its vicinity [16] is an idealized example of coupled solid-fluid systems interacting in the presence of vorticity and has applications to problems in engineering, such as locomotion of autonomous underwater vehicles, and in nature, such as swimming of fish. The first half of the paper presents simulation results and analysis of the system with $N=1$ and $N=2$, where the latter case is more relevant to the swimming of fish. The second half of this paper is devoted to theoretical analysis of the time-optimal control of the system. The control input is a bounded external force acting through the cylinder center-of-mass. The Maximum Principle and its later extensions, in particular those due to Sontag and Sussmann, are applied to this system for the case of the cylinder interacting with a single point vortex to arrive at conclusions regarding characteristics of the time-optimal controller.**

I. INTRODUCTION

The problem of locomotion of a solid body in a fluid flow endowed with vorticity is a fascinating problem both from a fluid mechanics viewpoint as well as a dynamics and control viewpoint. The problem has many applications ranging from the design of autonomous underwater vehicles that seek to conserve their available energy by utilizing the energy of vortices in their vicinity to the well-studied (though still not well-understood) problems of fish swimming and bird/insect flight. Traditional fluid mechanics approaches to these problems have, typically, either relied on experiments and numerical simulations or studied theoretical models where the solid body is fixed in place and expressions are obtained for the forces and moments on the body. Theoretical dynamics and control models, in particular finite-dimensional models, are far fewer. The point vortex model in theoretical fluid mechanics is a popular finite dimensional approximation of a fluid flow with coherent vortical structures [9], [10]. A Hamiltonian point vortex based model of a solid cylinder interacting with a vorticity field was constructed in [16] and [15]. Though the model obviously fails to address issues related to 3D effects and viscosity, which can be quite important in locomotion problems, it preserves the essential nonlinearities of the dynamics and provides a nice finite-dimensional model for studying the dynamics and control of interacting fluid-solid systems. The Hamiltonian structure, Poisson brackets and some dynamic features of this system have been investigated in the above references as also in [2] and [1].

In this paper we first present a few more interesting aspects of the control-free dynamics of this model, especially in relation to the locomotion problem, and then formulate a time-optimal control problem for this system, based on Pontryagin's maximum principle [12] and its extensions by Sontag and Sussmann [17], and present some results of our analysis of the singular nature of the controllers for the case $N=1$.

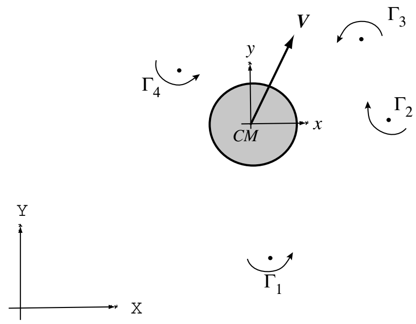

Fig. 1. N point vortices dynamically interacting with a 2-D rigid circular cylinder

II. THE CONTROL-FREE DYNAMICAL SYSTEM

A. Equations of motion

A schematic sketch of the model being considered is shown in Figure 1. The circular cylinder moves due to the non-uniform pressure field on its surface generated by the flow field of each of the vortices as well as the potential flow generated by its own motion. Simultaneously, each vortex is advected by the flow of all the other vortices as well as the potential flow generated by the cylinder motion. Free-slip boundary conditions prevail on the cylinder surface and hence rotations of the cylinder do not affect and are not affected by the dynamics of the system.

The equations of motion of this Hamiltonian model, as derived in [16] and [15], are:[1]

$$\frac{d\mathbf{L}}{dt} = -\mathbf{V}\times\Gamma\mathbf{k}, \tag{1}$$

$$\Gamma_j\frac{d\mathbf{l}_j}{dt} = -J\frac{\partial H}{\partial \mathbf{l}_j}, \quad j=1,\ldots,N, \tag{2}$$

[1] All quantities in the equations are with reference to a body-fixed frame whose origin is at the body center-of-mass.

where $\mathbf{V}$ is the velocity of the body center of mass, $\mathbf{l}_j$ is the position vector of the jth point vortex in the body-fixed frame, $\mathbf{k}$ is the unit vector normal to the plane, $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is the canonical symplectic matrix, Γ_j is the strength of the jth point vortex, $\Gamma = \sum_{j=1}^{N} \Gamma_j$ and $\mathbf{L}$ is the linear momentum of the system (i.e. fluid linear impulse plus cylinder linear momentum) given by:

$$\mathbf{L} = c\mathbf{V} + \sum_{j=1}^{N} \Gamma_j \mathbf{l}_j \times \mathbf{k} + R^2 \sum_{j=1}^{N} \mathbf{k} \times \Gamma_j \left(\frac{x_j}{x_j^2 + y_j^2}, \frac{y_j}{x_j^2 + y_j^2} \right). \tag{3}$$

In the above R is the radius of the cylinder and $c = m + \pi R^2$, where m is the mass and πR^2 is the "added mass". The Hamiltonian function H is given by:

$$\begin{aligned} H(\mathbf{L}, \mathbf{l}_j) = & -W(\mathbf{L}, \mathbf{l}_j) + \frac{1}{2c} \langle \mathbf{L}, \mathbf{L} \rangle - \frac{1}{c} \left(\sum_{j=1}^{N} \Gamma_j (\mathbf{L} \times \mathbf{l}_j) \cdot \mathbf{k} \right. \\ & - \frac{1}{2} \sum_{j=1}^{N} \Gamma_j^2 \langle \mathbf{l}_j, \mathbf{l}_j \rangle - \sum_{k=1}^{N} \sum_{j(j>k)}^{N} \Gamma_k \Gamma_j \langle \mathbf{l}_k, \mathbf{l}_j \rangle \\ & \left. + \frac{R^4}{2} \left\langle \sum_{j=1}^{N} \Gamma_j \frac{\mathbf{l}_j}{\langle \mathbf{l}_j, \mathbf{l}_j \rangle}, \sum_{j=1}^{N} \Gamma_j \frac{\mathbf{l}_j}{\langle \mathbf{l}_j, \mathbf{l}_j \rangle} \right\rangle \right). \end{aligned} \tag{4}$$

The Hamiltonian H is the body+fluid kinetic energy minus infinity terms, and this can be more easily seen by rewriting H in terms of $\mathbf{V}$ and $\mathbf{l}_j$; see [16] for details. The infinity terms are due to (i) blow-up of the velocity fields at the point vortex locations and (ii) the divergence of the kinetic energy integral as one goes to infinity for the case $\Gamma \neq 0$. The function W is related to the Kirchhoff-Routh function [7]. Note that in the above expression, as also in the rest of this paper, the notation $\langle \, , \rangle$ stands for the canonical inner-product on Euclidean space $\mathbb{R}^N$, viewed, also, as the canonical pairing between $\mathbb{R}^N$ and $\left(\mathbb{R}^N\right)^*$.

B. Trajectories for the case $N = 1$ and $N = 2$

We first present some simulations of cylinder and point vortex trajectories of system (1) and (2), for the cases $N = 1$ and $N = 2$. For all simulations in this paper, $R = 1.2$m and $m = 1$kg.

For $N = 1$ case, a representative graph of the trajectories of the interacting cylinder and point vortex is shown in Figure 2. The trajectories of the cylinder's center of mass and the point vortex are relative to a spatially fixed reference frame. The center of mass of the cylinder is initially at the origin of the spatially fixed reference frame. Γ_1=1m/s^2. Integrability of the $N = 1$ system has been shown in [2].

The case $N = 2$ is more relevant for the applications we have in mind, in particular for the swimming of fish that shed vortex pairs in their wake [18]. The most commonly observed vortex pattern is a counter-rotating pair shed in quick succession. In Figure 3, the simulation results of the trajectories of the cylinder's center of mass and two symmetric point vortices (i.e., $\Gamma_1 = -\Gamma_2$, $x_1(0) = x_2(0)$ and $y_1(0) = -y_2(0)$) are shown in a fixed reference frame. The center of mass of the cylinder is initially located at the

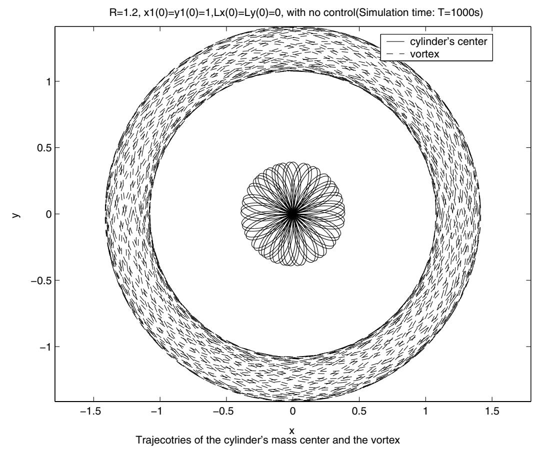

Fig. 2. Cylinder and point vortex trajectories for the case $N = 1$ with no controls.

origin of the fixed frame, and $\Gamma_1 = -\Gamma_2 = -1$m/s^2. Figure

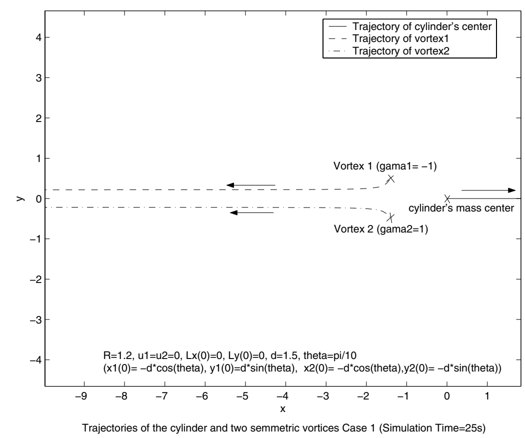

Fig. 3. Cylinder and trajectories of a symmetric counter-rotating pair of point vortices for the case $N = 2$ with no controls. $d^2 = x_1(0)^2 + y_1(0)^2 = x_2(0)^2 + y_2(0)^2$, $\theta = \arcsin(y_1(0)/d)$.

4 shows the corresponding velocity curves (in x direction) of the center of mass of the cylinder and the top point vortex, respectively, during the motion. The initial drop in the cylinder's velocity in Figure 4 is a consequence of the choice of the initial angular position of the vortices relative to the cylinder. It should be noted that in Figures 3 and 4, the initial condition $\mathbf{L}(0) = 0$ (and therefore $\mathbf{L}(t) = 0$ from (1)) was chosen to mimic the real situation where the swimming motion of the fish begins from a quiescent initial condition.

In the real situation of a swimming fish, the generation of the vortex pair is due to viscous effects. However, the subsequent forward thrust generated on the fish body is essentially a transfer of linear momentum phenomenon which, as these figures show, is captured even in this idealized model. In particular, it is interesting to observe the acceleration of the cylinder when the vortices are in close vicinity and the eventual loss of acceleration as the vortices move away from the cylinder. Viscous effects in the real case are of course likely to reduce these levels of acceleration and for sustained

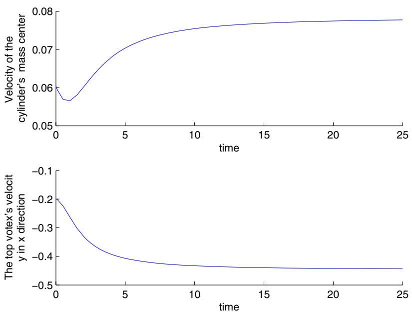

Fig. 4. Velocity of the mass center of the cylinder and the velocity in x direction of the top point vortex during the motion in Figure 3.

forward thrust it is obvious that the fish will need to keep shedding vortices.

Note that in Figures 3 and 4, the top vortex rotates clockwise and the bottom one counterclockwise. This should be contrasted with the *moving Föppl equilibrium* [6], [16], shown in Figure 5. In this translating equilibrium, where the whole configuration moves with a constant velocity $\mathbf{V}$ the symmetric counter-rotating pair of vortices in the cylinder wake has *opposite* signs to the pair in Figure 3.

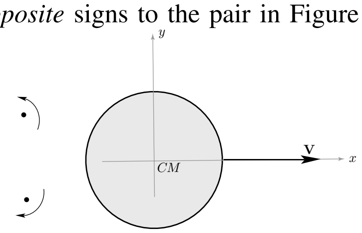

Fig. 5. Moving Föppl equilibrium,

III. The control system with an external force on the cylinder

In this section, we present the equations of motion of the system in the presence of a control input in the form of a force $\mathbf{F}$ acting through the center of mass of the cylinder, as shown (for the case $N=1$) in Figure 6.

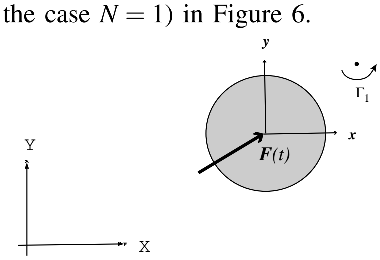

Fig. 6. The control system of a circular cylinder interacting with a single point vortex

A. Equations of motion

In the presence of a control input in the form of an external force $\mathbf{F}$ acting through the center of mass of the body($\equiv$ centroid $\equiv$ origin of body-fixed frame), the equations of motion (1) and (2) become:

$$\frac{d\mathbf{L}}{dt} = -\mathbf{V}\times\Gamma\mathbf{k}+\mathbf{F}, \tag{5}$$

$$\Gamma_j\frac{d\mathbf{l}_j}{dt} = -J\frac{\partial H}{\partial \mathbf{l}_j}, \quad j=1,\ldots,N,, \tag{6}$$

with $\mathbf{L}$ and H defined by (3) and (4), respectively.

B. The time-optimal control problem

The system (5) and (6) can be re-written in the form of a $2N+2$ dimensional affine control system [11]:

$$\frac{d\mathbf{X}}{dt} = \mathbf{f}(\mathbf{X})+\mathbf{G}(\mathbf{X})\mathbf{u}, \tag{7}$$

where the state vector $\mathbf{X} = (\mathbf{L},\mathbf{l}_1,...,\mathbf{l}_j)^T$, the drift vector field $\mathbf{f}(\mathbf{X}) = (-\mathbf{V}\times\Gamma\mathbf{k}, -\frac{1}{\Gamma_1}J\frac{\partial H}{\partial \mathbf{l}_1},\ldots,-\frac{1}{\Gamma_N}J\frac{\partial H}{\partial \mathbf{l}_N})^T = (f_1,f_2,...,f_{2N+2})^T$, and $\mathbf{G}=(\mathbf{g}_1,\mathbf{g}_2)$ where $\mathbf{g}_1=(1,0,...,0)^T$ and $\mathbf{g}_2=(0,1,0,...,0)^T$ are $2N+2$ dimensional vectors. The vector $\mathbf{u}=(u_1,u_2)^T$ is the control input vector with elements u_1 and u_2 being the components of the external force $\mathbf{F}$ in the x and y directions respectively. It is assumed that the control input vector is bounded, i.e., $u_{i\min}\le u_i \le u_{i\max}$, $u_{i\min}<0$, $u_{i\max}>0$, $i=1,2$.

Our goal is the standard point-to-point transfer problem, i.e., to find a bounded controller $\mathbf{u}(t)$ which moves the finite-dimensional system (7) from a given initial state $\mathbf{X}(t_1)=\mathbf{X}_i$ to a given final state $\mathbf{X}(t_2)=\mathbf{X}_f$ in the shortest time interval, i.e. that minimizes the cost functional

$$J=\int_{t_1}^{t_2} f_0(\mathbf{X},\mathbf{u})dt,$$

where $f_0=1$. A controller that achieves this goal is termed an optimal controller and is denoted by $\mathbf{u}^*(t)$.

Our goal here, however, is not to numerically find an optimal controller but, assuming they exist, to analyze the nature of these controllers for (7) and, in particular, to investigate conditions under which they are 'bang-bang', i.e. piecewise continuous, and when they are not. This analysis can then be used to design controllers and verify results of numerical simulations, which is one of our future working directions. The theoretical tools applied here are provided by Pontryagin's Maximum Principle [12] and theorems due to Sontag and Sussmann [17] which we briefly review in the next subsection. In addition to the time-optimal problem, we will also investigate the local accessibility of the system (7) and the accessibility algebra [11] for the case $N=1$.

C. Time-optimal control theory: Background

1) The Maximum Principle: The Maximum Principle [12] provides necessary conditions for the optimal controller. An optimal controller, $\mathbf{u}^*(t)$, which minimizes the cost functional J, maximizes the Pontryagin Hamiltonian

$$H_P(\mathbf{X},\lambda,\mathbf{u}) = \lambda^T\mathbf{f}(\mathbf{X})+\sum_{i=1}^{2}\lambda^T\mathbf{g}_i(\mathbf{X})u_i+\lambda_0 f_0 \tag{8}$$

with respect to $\mathbf{u}$ during the time interval $[t_1,t_2]$, where λ_0 is a non-positive constant. It follows that $\partial H_P/\partial\mathbf{u}=0$ for $\mathbf{u}=\mathbf{u}^*(t)$ and $t\in[t_1,t_2]$. The adjoint vector λ in the Hamiltonian is defined in such a way that the components of the adjoint vector λ and the state variable vector $\mathbf{X}$ form a canonical Hamiltonian system:

$$\dot{x}_j = \frac{\partial H_P}{\partial\lambda_j}, \quad \dot{\lambda}_j = -\frac{\partial H_P}{\partial x_j}, \quad j=1,...,2N+2, \tag{9}$$

and $\lambda(t) \neq 0$, $t \in [t_1, t_2]$.

If an optimal controller $\mathbf{u}^*(t)$ can be found then one can, in principle, solve (9) and find the corresponding state and costate trajectories. A triple $(\mathbf{X}^*(t), \mathbf{u}^*(t), \lambda^*(t))$ satisfying the Maximum Principle is defined as an extremal. In an affine control problem, such as this one, it is obvious that on a time interval where the coefficient of u_i in the Pongtryagin Hamiltonian, $\lambda^T \mathbf{g}_i$, is not zero, then u_i must take its bounded value, $u_{i\max}$ or $u_{i\min}$, to maximize H_P during this time interval. This mode is the so-called 'bang-bang' control. Since the choice of $u_{i\max}$ or $u_{i\min}$ depends on the sign of $\phi_i = \lambda^T \mathbf{g}_i$, ϕ_i is defined as 'switching function'. However, there are also cases when $\lambda^T \mathbf{g}_i = 0$ on a time interval, and these are defined as 'u_i-singular' cases [17]. It is to this topic we turn in the next paragraph.

2) The Sontag-Sussmann Theorems: In [17], Sontag and Sussmann presented a series of theorems for the singular extremal cases. The statement of two of these, to be applied to our model, are given below.

The notation used is standard. The Lie bracket $[\mathbf{a}, \mathbf{b}]$ of vector fields $\mathbf{a}$ and $\mathbf{b}$ is: $[\mathbf{a}, \mathbf{b}] = D(\mathbf{b})\mathbf{a} - D(\mathbf{a})\mathbf{b}$, where $D()$ is the Jacobian, and higher order Lie brackets are denoted by $\mathrm{ad}^n_{\mathbf{a}} \mathbf{b} = [\mathbf{a}, [\mathbf{a}, \ldots [\mathbf{a}, \mathbf{b}] \ldots]]$, where there are $n \in \mathbb{Z}^+$ iterated brackets.

Theorem 3.1: **(Sontag-Sussmann):** Consider an n-dimensional system in form of (7) where $n = 2m$ and m is the dimension of the control input vector. In a nonempty time interval $[t_1, t_2]$, if

(i) $[\mathbf{g}_i, \mathbf{g}_j] = 0$, $i, j \in \{1, \ldots m\}$;
(ii) $\mathbf{g}_1, \ldots, \mathbf{g}_m, [\mathbf{f}, \mathbf{g}_1], \ldots, [\mathbf{f}, \mathbf{g}_m]$ are linearly independent

then there is no sub-interval in $[t_1, t_2]$ in which an extremal $(\mathbf{X}^*(t), \mathbf{u}^*(t), \lambda^*(t))$ is u_i-singular for all i, $i = 1, \ldots, m$, i.e. 'totally-singular'[4]. In this case, if an extremal is u_i-singular for all $i \neq k$, then u_k must be non-singular and thus 'bang-bang'.

Theorem 3.2: **(Sontag-Sussman):** Furthermore, u_k is constant, i.e. without switching, if

(i) $[\mathbf{g}_i, [\mathbf{f}, \mathbf{g}_j]] = 0$, $i, j \in \{1, \ldots, m\}$;
(ii) The vectors in the set $\{\mathbf{g}_i, i = 1, \ldots, m\} \cup \{[\mathbf{f}, \mathbf{g}_i], i = 1, \ldots, m, i \neq k\} \cup \{\mathrm{ad}^2_{\mathbf{f}} \mathbf{g}_i, i = 1, \ldots, m, i \neq k\}$ are linearly independent.

In our problem, the $N = 1$ case corresponds to the $n = 2m$ case discussed in the two theorems ($n = 2N + 2 = 4$ and $m = 2$). For $N \geq 2$, we have $n > 2m$ and the theorems need to be extended for applications. More detailed discussions of the $n > 2m$ cases can be found in Chyba et al.([4]), ([5]).

IV. Results for the case $N = 1$

The above theorems will now be applied to our model for the case $N = 1$. It should be noted that the Sontag-Sussmann theorems have been applied in a different model of a solid body moving in a fluid environment by Chyba et al [4], [5] who use Kirchhoff's equations of motion [9] in which the fluid flow is considered irrotational i.e. without vorticity. The presence of vorticity, even in the form of a simple point-vortex approximation as in our model, considerably complicates the fluid vector field. To apply the theorems to the general N case becomes a formidable computational task and therefore we focus, at least for this paper, on the $N = 1$ case. In addition to the time-optimal problem, we will also investigate the local accessibility of the system and the accessibility algebra [11].

A. The case $N = 1$

Consider the one cylinder with one point vortex case ($N = 1$). In this case, (7) becomes a four dimension system in which $\mathbf{X} = (L_x, L_y, x_1, y_1)^T$, where the pairs L_x, L_y and x_1, y_1 are the components of $\mathbf{L}$ and $\mathbf{l_1}$, respectively. The drift vector field is

$$\begin{aligned} \mathbf{f}(\mathbf{X}) &= \begin{pmatrix} f_1(\mathbf{X}) & f_2(\mathbf{X}) & f_3(\mathbf{X}) & f_4(\mathbf{X}) \end{pmatrix}^T \\ &= \begin{pmatrix} -\Gamma_1 V_y & \Gamma_1 V_x & \frac{1}{\Gamma_1}\frac{\partial H}{\partial y_1} & -\frac{1}{\Gamma_1}\frac{\partial H}{\partial x_1} \end{pmatrix}^T. \end{aligned}$$

1) Local Accessibility and Accessibility Algebra: Local accessibility gives information about the directions in which the system can be steered at each point in state space. The local accessibility of a nonlinear affine control system in the form of (7) depends on the dimension of the so-called accessibility algebra, in which every element is a linear combination of iterated Lie brackets of the form

$$[\mathbf{X}_k, [\mathbf{X}_{k-1}, [\ldots, [\mathbf{X}_2, \mathbf{X}_1] \ldots]]]$$

where each $\mathbf{X}_i$ is in the set $\{\mathbf{f}, \mathbf{g}_1, \ldots \mathbf{g}_m\}$, and $k = 1, 2, \ldots$. If the dimension of the span of these elements is equal to the dimension of the system at a given point in the state space, then the system is called locally accessible from that point [11].

In our problem for the $N = 1$ case, we have the following lemma:

Lemma 4.1: The system (7), with $N = 1$, is locally accessible at each $p \in P$.

Proof: For $N = 1$, consider the following four elements of the accessibility algebra: $\mathbf{g}_1, \mathbf{g}_2, [\mathbf{f}, \mathbf{g}_1]$ and $[\mathbf{f}, \mathbf{g}_2]$, and the matrix $E(p)$ comprising of the four column vectors $\mathbf{g}_1(p), \mathbf{g}_2(p), [\mathbf{f}, \mathbf{g}_1](p), [\mathbf{f}, \mathbf{g}_2](p)$. By calculation, one gets

$$\begin{aligned} \det(E(p)) &= \frac{1}{\Gamma_1^2}\left(\frac{\partial^2 H}{\partial y_1 \partial L_y}\frac{\partial^2 H}{\partial x_1 \partial L_x} - \frac{\partial^2 H}{\partial x_1 \partial L_y}\frac{\partial^2 H}{\partial y_1 \partial L_x}\right) \\ &= \frac{(x_1^2 + y_1^2)^2 - R^4}{c^2 (x_1^2 + y_1^2)^2}. \end{aligned}$$

Since the vortex cannot collide with the cylinder surface, $\det(E(p)) \neq 0$ at each $p \in P$. It follows that the four vector fields $\mathbf{g}_1, \mathbf{g}_2, [\mathbf{f}, \mathbf{g}_1], [\mathbf{f}, \mathbf{g}_2]$ are (pointwise) linearly independent, and thus the system is locally accessible at each point in the state space P. ■

2) Time-optimal controllers: Our results for the non-singular nature of the time-optimal controllers in the case $N = 1$ are stated in the following two lemmas:

Lemma 4.2: For system (7), with $N = 1$, if an extremal is u_i-singular on $[t_1, t_2]$, then the remaining one, u_k, $k \neq i$, must be nonsingular and 'bang-bang' on this time interval. Further, there exist subsets $A_i \subset P$, $i \in \{1, 2\}$, such that if the system state trajectory $\mathbf{x}(t) \in A_i$ for all $t \in [t_1, t_2]$, then the non-singular control input u_k ($k \neq i$) must be constant (without switching) in this time interval.

Proof: Apply Theorem 1 (Sontag-Sussmann), using the computation of the previous lemma and the trivial observation that $[\mathbf{g}_1, \mathbf{g}_2] = 0$, to obtain the first part of the Proposition.

Next, by calculation it is readily checked that $[\mathbf{g}_i, [\mathbf{f}, \mathbf{g}_j]] = 0$, $i, j \in \{1, 2\}$, and so the first part of Theorem 2 (Sontag-Sussmann) is satisfied.

Defining matrices $E_i(p)$=$[\mathbf{g}_1(p), \mathbf{g}_2(p), [\mathbf{f}, \mathbf{g}_i](p), \mathrm{ad}^2_{\mathbf{f}}\, \mathbf{g}_i(p)]$ for $i = 1, 2$, by calculation we obtain that $\det(E_1(p)) = S_1/w$ and $\det(E_2(p)) = S_2/w$, where

$$
\begin{aligned}
S_1(L_y, x_1, y_1) &= -4L_y x_1 (R^2 - x_1^2 - y_1^2)(x_1^2 + y_1^2) \\
&\quad + R^2 \Gamma_1 (-5x_1^4 - 4x_1^2 y_1^2 + y_1^4 + R^2(5x_1^2 + y_1^2)), \\
S_2(L_x, x_1, y_1) &= -4L_x y_1 (-R^2 + x_1^2 + y_1^2)(x_1^2 + y_1^2) \\
&\quad + R^2 \Gamma_1 (-5y_1^4 - 4x_1^2 y_1^2 + x_1^4 + R^2(x_1^2 + 5y_1^2)), \\
w(x_1, y_1) &= 8\pi^3 R^2 (x_1^2 + y_1^2)^5,
\end{aligned}
$$

and in which we have used the relation c=$m + \pi R^2$=$2\pi R^2$ which holds since the mass m of the cylinder is actually the mass per unit length (perpendicular to plane) and we take ρ=1 everywhere.

Thus, we can define subsets A_i where the vector fields $\mathbf{g}_1, \mathbf{g}_2, [\mathbf{f}, \mathbf{g}_i]$ and $\mathrm{ad}^2_{\mathbf{f}}\, \mathbf{g}_i$ are linearly independent given explicitly by $A_i = P \backslash S_i$, where S_i is the surface where the four vectors are linearly dependent (i.e., $\det(E_i(p)) = 0$):

$$
S_1(L_y, x_1, y_1) = 0; \quad S_2(L_x, x_1, y_1) = 0.
$$

■

Before stating the next lemma, we need to make some definitions. Recall that $u_{k\min} \leq u_k \leq u_{k\max}$, $u_{k\min} < 0$, $u_{k\max} > 0$, $k = 1, 2$. Now define the sets

$$
B_k = B_{k\max} \cup B_{k\min}, \quad k = 1, 2,
$$

where

$$
\begin{aligned}
B_{k\max} := \big\{ p \in P \mid\, & \mathbf{g}_i(p), [\mathbf{f}, \mathbf{g}_i](p), \mathrm{ad}^2_{\mathbf{f}}\, \mathbf{g}_i(p), \\
& \mathrm{ad}^3_{\mathbf{f}}\, \mathbf{g}_i(p) + u_{k\max} [\mathbf{g}_k, \mathrm{ad}^2_{\mathbf{f}}\, \mathbf{g}_i](p) \text{ are linearly independent} \big\},
\end{aligned}
$$

$$
\begin{aligned}
B_{k\min} := \big\{ p \in P \mid\, & \mathbf{g}_i(p), [\mathbf{f}, \mathbf{g}_i](p), \mathrm{ad}^2_{\mathbf{f}}\, \mathbf{g}_i(p), \\
& \mathrm{ad}^3_{\mathbf{f}}\, \mathbf{g}_i(p) + u_{k\min} [\mathbf{g}_k, \mathrm{ad}^2_{\mathbf{f}}\, \mathbf{g}_i](p) \text{ are linearly independent} \big\}
\end{aligned}
$$

for $k = 1, 2$ and $i \neq k$. Next, we show that

Lemma 4.3: $B_1 \cap B_2 \neq \phi$. Moreover, locally $B_1 \cap B_2$ has the same manifold structure as P.

Proof: The proof essentially relies on the fact that each $B_k = P \backslash W_k$, where W_k is the subset of P on which the four vector fields being considered are (pointwise) linearly *dependent*. With $k = 1$, for example, $W_1 = W_{1\max} \cup W_{1\min}$ where

$$
\begin{aligned}
W_{1\max} := \big\{ p \in P \mid \det \big(& \mathbf{g}_i(p), [\mathbf{f}, \mathbf{g}_i](p), \mathrm{ad}^2_{\mathbf{f}}\, \mathbf{g}_i(p), \\
& \mathrm{ad}^3_{\mathbf{f}}\, \mathbf{g}_i(p) + u_{1\max} [\mathbf{g}_k, \mathrm{ad}^2_{\mathbf{f}}\, \mathbf{g}_i](p) \big) = 0 \big\},
\end{aligned}
$$

$$
\begin{aligned}
W_{1\min} := \big\{ p \in P \mid \det \big(& \mathbf{g}_i(p), [\mathbf{f}, \mathbf{g}_i](p), \mathrm{ad}^2_{\mathbf{f}}\, \mathbf{g}_i(p), \\
& \mathrm{ad}^3_{\mathbf{f}}\, \mathbf{g}_i(p) + u_{1\min} [\mathbf{g}_k, \mathrm{ad}^2_{\mathbf{f}}\, \mathbf{g}_i](p) \big) = 0 \big\},
\end{aligned}
$$

and, similarly, $W_2 = W_{2\max} \cup W_{2\min}$.

If both W_1 and W_2 are empty, then $B_1 = B_2 = P$ and the lemma is obviously proved. Consider now the case where one or both of W_1 and W_2 non-empty. It can be shown (see Appendix) that each of the sets $W_{1\max}, W_{1\min}, W_{2\max}$ and $W_{2\min}$ can be characterized as level sets of certain functions on P. In other words, they consist of points, $p \equiv (L_x, L_y, x_1, y_1) \in P$, which satisfy equations of the following form, respectively,

$$
Q_1 = u_{1\max}; \quad Q_1 = u_{1\min}; \quad Q_2 = u_{2\max}; \quad Q_2 = u_{2\min}, \tag{10}
$$

where Q_1 and Q_2 are smooth rational functions of L_x, L_y, x_1, y_1, i.e they are of the form

$$
Q_1 = \frac{p_1(L_x, L_y, x_1, y_1)}{q_1(L_x, L_y, x_1, y_1)}; \quad Q_2 = \frac{p_2(L_x, L_y, x_1, y_1)}{q_2(L_x, L_y, x_1, y_1)},
$$

where p_1, p_2, q_1 and q_2 are polynomial functions. Away from the zeroes of q_1 and q_2, the functions Q_1 and Q_2 will therefore be well-defined and equations like(10) will locally define smooth manifolds of dimension less than or equal to 3. Generally speaking therefore each of W_1 and W_2, and hence $W_1 \cup W_2$, will be a piecewise smooth manifold (possibly with boundary and possibly not connected) of dimension less than or equal to 3 . Therefore, $B_1 \cap B_2 = \{P \backslash W_1\} \cap \{P \backslash W_2\} = P \backslash \{W_1 \cup W_2\} \neq \phi$ and locally $B_1 \cap B_2$ will have the same manifold structure as P. ■

We are now ready to state our final lemma:

Lemma 4.4: For system (7), with $N = 1$, if the system state optimal trajectory $\mathbf{x}(t) \in B_1 \cap B_2 \neq \phi$ for all $t \in [t_1, t_2]$, then both u_1 and u_2 must necessarily be nonsingular in this time interval.

Proof: The proof is by contradiction. Assume that the extremal is u_1-singular on $[t_1, t_2]$. By definition and computation, we have the following four equations:

$$
\begin{aligned}
&\phi_1 = \lambda^T \mathbf{g}_1 = 0;\ \phi_1' = \lambda^T [\mathbf{f}, \mathbf{g}_1] = 0;\ \phi_1'' = \lambda^T \mathrm{ad}^2_{\mathbf{f}}\, \mathbf{g}_1 = 0; \\
&\phi_1''' = \lambda^T (\mathrm{ad}^3_{\mathbf{f}}\, \mathbf{g}_1 + u_2 [\mathbf{g}_2, \mathrm{ad}^2_{\mathbf{f}}\, \mathbf{g}_1]) = 0,
\end{aligned}
$$

where $'$ denotes time differentiation. Note that since u_2 is piecewise constant by Lemma 4.2, $u_2 = \{u_{2\min}, u_{2\max}\}$. By definition, $\mathbf{g}_1(p)$, $[\mathbf{f}, \mathbf{g}_1](p)$, $\mathrm{ad}^2_{\mathbf{f}}\, \mathbf{g}_1(p)$ and $\mathrm{ad}^3_{\mathbf{f}}\, \mathbf{g}_1 + u_2 [\mathbf{g}_k, \mathrm{ad}^2_{\mathbf{f}}\, \mathbf{g}_1](p)$ are linearly independent for *all* $p \in B_2$, and hence for *all* $p \in B_1 \cap B_2$. Therefore, the only solution to the above four equations in $B_1 \cap B_2$ is $\lambda = \mathbf{0}$ which contradicts that $\lambda \neq \mathbf{0}$ in the Pontryagin Hamiltonian. Hence, the assumption of u_1-singular cannot hold. Similarly, the assumption of u_2-singular leads to the same contradiction. Therefore, if $\mathbf{x}(t)$ lives in $B_1 \cap B_2$ for all t in $[t_1, t_2]$, then the extremal must necessarily be nonsingular for both u_1 and u_2

■

V. CONCLUSION REMARKS

In this paper we have presented some dynamical features of a Hamiltonian model of a circular cylinder interacting with N point vortices and also preliminary results on the time-optimal control of this problem for the case $N=1$. In the near future, we hope to extend these investigations in the following directions:

(i) $N \geq 2$ *cases:* As demonstrated in the dynamics part of this paper, special configurations of the case $N=2$ clearly show the benefits of vorticity for the locomotion of the cylinder. The time-optimal control analysis for $N=2$ however is much more tedious than for $N=1$ due to the higher dimension of the system. However, we hope to compute such controllers numerically.

(ii) *Other optimality criteria and control inputs:* Total time of travel may not necessarily be the appropriate cost function to optimize in a locomotion problem. There are other equally relevant cost functions such as, for example, the L^2-norm of the controls. Optimizing the mechanical energy in this model is also interesting due to the fact that the control-free system is a Hamiltonian system which conserves mechanical energy. Moreover, instead of having force inputs, we will also consider the shape as a control input and self-propulsion by shape changes in the presence of vorticity, analogous to the work of Shapere and Wilczek [14] in the low Reynolds number limit.

(iii) *Symmetry reduction and Hamiltonian structure:* Given the underlying Hamitlonian structure of the control-free system, another direction to pursue is to exploit the Hamiltonian structure for our control goals. In this regard, we expect works like [3] to be useful.

APPENDIX

The sets W_1 and W_2, used in Lemma 4.3, can be characterized as follows. Take W_2, for example. By calculation one obtains the vector fields

$$[\mathbf{f},\mathbf{g}_1] = \begin{bmatrix} 0 & -\frac{\Gamma_1}{c} & \frac{R^2(y_1^2-x_1^2)}{c(x_1^2+y_1^2)^2}+\frac{1}{c} & -\frac{2R^2x_1y_1}{c(x_1^2+y_1^2)^2} \end{bmatrix}^T,$$

$$\mathrm{ad}_{\mathbf{f}}^2\,\mathbf{g}_1 = \begin{bmatrix} -\frac{R^4\Gamma_1^2}{c^2(x_1^2+y_1^2)^2} & 0 & C_3 & C_4 \end{bmatrix}^T,$$

$$\mathrm{ad}_{\mathbf{f}}^3\,\mathbf{g}_1 = \begin{bmatrix} D_1 & D_2 & D_3 & D_4 \end{bmatrix}^T,$$

where the entries C_3, C_4, D_1, D_2, D_3 and D_4 are rational, nonlinear functions of L_x, L_y, x_1 and y_1 and whose explicit form is not needed for our purpose here.

It follows from the above that

$$[\mathbf{g}_2,\mathrm{ad}_{\mathbf{f}}^2\,\mathbf{g}_1] = \begin{bmatrix} 0 & 0 & -\frac{4R^4y_1}{c^2(x_1^2+y_1^2)^3} & \frac{4R^4x_1}{c^2(x_1^2+y_1^2)^3} \end{bmatrix}^T,$$

Consider the matrix

$$\mathbf{N}_{\max} = \begin{pmatrix} \mathbf{g}_1 & [\mathbf{f},\mathbf{g}_1] & \mathrm{ad}_{\mathbf{f}}^2\,\mathbf{g}_1 & \mathrm{ad}_{\mathbf{f}}^3\,\mathbf{g}_1 + u_{2\max}[\mathbf{g}_2,\mathrm{ad}_{\mathbf{f}}^2\,\mathbf{g}_1] \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 & -\frac{R^4\Gamma_1^2}{c^2(x_1^2+y_1^2)^2} & D_1 \\ 0 & -\frac{\Gamma_1}{c} & 0 & D_2 \\ 0 & \frac{R^2(y_1^2-x_1^2)}{c(x_1^2+y_1^2)^2}+\frac{1}{c} & C_3 & D_3 - u_{2\max}\frac{4R^4y_1}{c^2(x_1^2+y_1^2)^3} \\ 0 & -\frac{2R^2x_1y_1}{c(x_1^2+y_1^2)^2} & C_4 & D_4 + u_{2\max}\frac{4R^4x_1}{c^2(x_1^2+y_1^2)^3} \end{pmatrix}$$

Similarly, define $\mathbf{N}_{\min}$. By definition,

$$W_2 := \{p \in P \mid \det(\mathbf{N}_{\max}(p)) = 0\} \cup \{p \in P \mid \det(\mathbf{N}_{\min}(p)) = 0\}$$

From the structure of the matrices, it is easy to see that the equations $\det(\mathbf{N}_{\max}(p)) = 0$ and $\det(\mathbf{N}_{\min}(p)) = 0$ can be rewritten as, respectively,

$$Q_2(L_x,L_y,x_1,y_1) = u_{2\max}; \quad Q_2(L_x,L_y,x_1,y_1) = u_{2\min}$$

where Q_2 is a smooth rational function of its arguments i.e. it is of the form

$$Q_2(L_x,L_y,x_1,y_1) = p_2(L_x,L_y,x_1,y_1)/q_2(L_x,L_y,x_1,y_1),$$

where p_2 and q_2 are polynomial functions. Thus, W_2 can be characterized as the union of level sets of function Q_2 (wherever defined) corresponding to values $u_{2\max}$ and $u_{2\min}$. Similarly, Q_1 and W_1.

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