J Nonlinear Sci
DOI 10.1007/s00332-009-9051-4

Journal of Nonlinear Science

Dissipative N-point-vortex Models in the Plane

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Received: 8 September 2008 / Accepted: 16 July 2009

Abstract A method is presented for constructing point vortex models in the plane that dissipate the Hamiltonian function at any prescribed rate and yet conserve the level sets of the invariants of the Hamiltonian model arising from the SE(2) symmetries. The method is purely geometric in that it uses the level sets of the Hamiltonian and the invariants to construct the dissipative field and is based on elementary classical geometry in $\mathbb{R}^3$. Extension to higher-dimensional spaces, such as the point vortex phase space, is done using exterior algebra. The method is in fact general enough to apply to any smooth finite-dimensional system with conserved quantities, and, for certain special cases, the dissipative vector field constructed can be associated with an appropriately defined double Nambu–Poisson bracket. The most interesting feature of this method is that it allows for an infinite sequence of such dissipative vector fields to be constructed by repeated application of a symmetric linear operator (matrix) at each point of the intersection of the level sets.

Keywords Dissipative · Point vortex · Symmetries · Hamiltonian

Mathematics Subject Classification (2000) 76B47 · 34C14 · 37C99

Communicated by C. Rowley.

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Published online: 10 October 2009

1 Introduction

The N-point-vortex model in the unbounded plane and other topological surfaces is a well-established area of research in the framework of Euler's equations for a classical fluid flow, see Newton (2001) or Saffman (1992) for good surveys on the subject.

In the plane the equations of motion of N point vortices, with coordinates $(x_j, y_j) \equiv \mathbf{l}_j$, $j = 1, \ldots, N$, has the Hamiltonian structure

$$\frac{d\mathbf{l}_j}{dt} = J_{\text{pv}} \nabla_{\mathbf{l}_j} H_{\text{pv}} =: X_{\text{HPV}}, \tag{1}$$

where

$$J_{\text{pv}} = \begin{pmatrix} 0 & \frac{1}{\Gamma_1} & 0 & \cdot & \cdot & 0 \\ -\frac{1}{\Gamma_1} & 0 & 0 & \cdot & \cdot & 0 \\ 0 & 0 & 0 & \frac{1}{\Gamma_2} & \cdot & 0 \\ 0 & 0 & -\frac{1}{\Gamma_2} & 0 & \cdot & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{\Gamma_N} \\ 0 & 0 & 0 & 0 & -\frac{1}{\Gamma_N} & 0 \end{pmatrix}, \tag{2}$$

Γ_j denoting the strength of the jth point vortex, and the kinetic energy Hamiltonian function, conserved by the dynamics, is

$$H_{\text{pv}} = -\frac{1}{4\pi} \sum_{i}^{N} \sum_{j=i+1}^{N} \Gamma_i \Gamma_j \log |\mathbf{l}_i - \mathbf{l}_j|^2. \tag{3}$$

The Hamiltonian function is invariant under rigid body translations and rotations, i.e., under the action of the Lie group SE(2), and by Noether's theorem this leads to three more conserved quantities of the motion (up to constant multiplicative factors)

$$I_1 = \sum_{i=1}^{N} \Gamma_i x_i, \qquad I_2 = \sum_{i=1}^{N} \Gamma_i y_i, \qquad I_3 = \sum_{i=1}^{N} \Gamma_i \left(x_i^2 + y_i^2\right). \tag{4}$$

The above three invariants are of course just a special case of the six invariants of motion of the incompressible Euler equations in $\mathbb{R}^3$ (with a sufficiently rapid decay of vorticity at infinity to ensure the convergence of all integrals)

$$\mathbf{L} = \int_{\mathbb{R}^3} \mathbf{r} \times \omega \, dV, \qquad \mathbf{A} = \int_{\mathbb{R}^3} |\mathbf{r}|^2 \omega \, dV, \tag{5}$$

where $\mathbf{L}$, $\mathbf{A}$, $\omega \in \mathbb{R}^3$, and $\mathbf{r}$ is the position vector in $\mathbb{R}^3$.

It is also true, however, that (5) are conserved by the incompressible Navier–Stokes equations in $\mathbb{R}^3$, despite the fact that the kinetic energy (in the absence of any external forcing) decays.[1] In the Western scientific literature, this discovery is

[1] In $\mathbb{R}^2$, the quantity $\mathbf{A}$ is conserved only if the total vorticity $\int_{\mathbb{R}^2} \omega \, dV$ is zero; a result going as far back as Poincaré (1893). The methodology of our paper can be applied to the case of nonzero total vorticity as well.

attributed to Moreau; see the books by Truesdell (1954) and Ting and Klein (1991) for nice discussions on the topic. In the presence of fixed boundaries, the SE(3) (or SE(2)) symmetries are lost, and the conservation properties of (5) may no longer persist.

The principal motivation of this paper is to come up with an N-point-vortex model that retains the above features of the Navier–Stokes equations in unbounded domains, i.e., the model should

Feature 1 Dissipate the kinetic energy Hamiltonian (3).

Feature 2 Conserve the quantities (4).

We present a method in this paper for constructing such models which is in fact general enough to be applicable to any smooth finite-dimensional system with conserved quantities (not necessarily those associated with symmetries). Further, the rate of dissipation of the Hamiltonian function can be prescribed arbitrarily. But, perhaps most interestingly, the method does the following:

> It defines a symmetric linear operator at each point of the intersecting level sets that leaves invariant the "category" of smooth vector fields with Features 1 and 2.

In other words, an infinite set of vector fields with the above two features can be constructed.

The idea of adding dissipation to general Hamiltonian systems is of course not new, and, in particular, Brockett's idea of double bracket dissipation (Brockett 1991, 1993, 1994) has led to the development of several elegant methods. In the context of the Euler–Poincaré equations, one such method is the one developed by Bloch et al. (1996) in great detail. Indeed the germ of the idea underlying our method may be discernible in the work of Bloch et al. (1996) (see the section "Motivating Examples"). However, their method of constructing dissipative fields, though with the same goal in principle, is different from ours. There the focus is on constructing double bracket dissipative fields in the framework of cotangent bundles of Lie groups and their symmetry reduced spaces which are identified with the dual of the Lie algebra. Euler–Poincaré and Lie–Poisson formulations arise naturally in this framework, and the dissipative fields constructed conserve the coadjoint orbits of the dual of the Lie algebra. In our method, knowledge of these geometric mechanics concepts is not necessary, and the underlying idea is elementary and intuitive. In addition, the point vortex model has certain peculiarities which suggest that the manner of application of the Bloch et al. method to this model may not be entirely obvious, namely, (i) the Hamiltonian point vortex model phase space is not a cotangent bundle, (ii) there is no "nice" associated Euler–Lagrange or Euler–Poincaré formulation for the Hamiltonian point vortex equations, and (iii) the dissipative fields that are sought need to conserve the SE(2) symmetries in addition to coadjoint orbit preservation (Marsden and Weinstein 1983). But, perhaps more importantly, it is not clear if double bracket dissipative fields lend themselves to defining an invariant "category" of such fields. Nevertheless, for certain special cases, the general methodology presented in this paper can be interpreted in terms of double bracket dissipation via the Nambu–Poisson brackets (Nambu 1973). Other important works on double bracket dissipation include the paper by Holm et al. (2008), which introduces a modified form of double bracket dissipation, the general case of double bracket dissipation analyzed in the paper by Bloch et al. (1992), and the general case of optimal double bracket flows analyzed in Bloch et al. (1997).

Papers on dissipative point vortex models in the scientific literature seem to be few and sparse. The work by Scobelev and Shmagunov (1998), with an emphasis on numerical vortex methods, is the earliest one known to the author. Holm et al. also use their double bracket dissipation technique to construct a point vortex model. Discussions on a dissipative point vortex model can also be found in the recent book by Kozlov (2008); see also the papers by Agullo and Verga (1997, 2001). Papers on numerical vortex methods, alluded to previously, are far more; see Leonard (1985) and Cottet and Koumoutsakos (2000) for reviews. In such methods, pioneered by Chorin (1973, 1978), the vorticity field is regularized and supported on blobs of finite size. Dissipative and diffusive effects are typically modeled by incorporating random walk schemes. The motivation for these methods arises primarily from the need for developing efficient numerical schemes for the Navier–Stokes equations.

As in the double bracket setting, the construction of the dissipative total vector field X_D in our model is done affinely, i.e., by the addition of a dissipative vector field X_d to the Hamiltonian vector field X_H:

$$X_D = X_H + X_d. \tag{6}$$

The outline of this paper is as follows. In Sect. 2, we describe the method in detail in the context of general smooth finite-dimensional vector fields. In Sect. 3, we discuss the relation to Nambu–Poisson brackets. In Sect. 4, we apply the method to the two-point-vortex and three-point-vortex models. In Sect. 5, we summarize, discuss, and point out future directions of research. The mathematical background required for Sects. 2 and 3 and the computations involved in the models of Sect. 4 are presented in three appendices at the end.

2 Dissipative Vector Fields Conserving Level Sets of Real-Valued Functions in Euclidean Spaces

In this section the method is described in the general setting of smooth vector fields on finite-dimensional spaces.

Consider first $\mathbb{R}^3$ with the standard Euclidean inner product, denoted by $\langle\, , \rangle$, and smooth functions $C, H : \mathbb{R}^3 \to \mathbb{R}$. A straightforward application of the scalar triple product rule then shows that the vector field

$$-\nabla C \times (\nabla H \times \nabla C) \tag{7}$$

has the following two properties:

(a) $-\langle \nabla H, \nabla C \times (\nabla H \times \nabla C)\rangle \leq 0$,
(b) $-\langle \nabla C, \nabla C \times (\nabla H \times \nabla C)\rangle = 0$.

Since ∇C is normal to the level set of C through any point (provided that $\nabla C \neq 0$), it follows that the vector field (7) conserves the level sets of C while at the same time dissipating the function H. This remarkably simple idea is the basis of the construction of the dissipative fields X_d in this paper.

If we now consider three smooth functions $C_1, C_2, H : \mathbb{R}^3 \to \mathbb{R}$, then the vector field

$$-(\nabla C_1 \times \nabla C_2) \times \big(\nabla H \times (\nabla C_1 \times \nabla C_2)\big) \tag{8}$$

can be shown, again using the scalar triple product, to have the following two properties:

(a) $-\langle \nabla H, (\nabla C_1 \times \nabla C_2) \times (\nabla H \times (\nabla C_1 \times \nabla C_2))\rangle \leq 0$,
(b) $-\langle \nabla C_i, (\nabla C_1 \times \nabla C_2) \times (\nabla H \times (\nabla C_1 \times \nabla C_2))\rangle = 0,\ i = 1, 2$,

implying that (8) conserves the level sets of C_1 and C_2 while dissipating H.

To apply the above concepts in a higher-dimensional Euclidean space, we need to generalize the concepts of cross-products and triple products, and this naturally leads to wedge products. Basic exterior algebra concepts used in the derivations below are introduced in Appendix A. Equip the space $\mathbb{R}^K$, $K > 3$, with the standard Euclidean inner product denoted by $\langle\,,\,\rangle$ and consider the vector field given by

$$-\big(*\big(\mathbf{d}C \wedge *(\mathbf{d}H \wedge \mathbf{d}C)\big)\big)^\sharp, \tag{9}$$

where $C, H : \mathbb{R}^K \to \mathbb{R}$. Now, from (72) and from the properties of the wedge product we obtain

$$\begin{aligned}
&-\big\langle (\mathbf{d}C)^\sharp, \big(*\big(\mathbf{d}C \wedge *(\mathbf{d}H \wedge \mathbf{d}C)\big)\big)^\sharp\big\rangle \\
&\quad = - * \big(\mathbf{d}C \wedge * * \big(\mathbf{d}C \wedge *(\mathbf{d}H \wedge \mathbf{d}C)\big)\big) \\
&\quad = (-1)^K * \big(\mathbf{d}C \wedge \big(\mathbf{d}C \wedge *(\mathbf{d}H \wedge \mathbf{d}C)\big)\big) \\
&\quad = 0
\end{aligned} \tag{10}$$

and

$$\begin{aligned}
&-\big\langle (\mathbf{d}H)^\sharp, \big(*\big(\mathbf{d}C \wedge *(\mathbf{d}H \wedge \mathbf{d}C)\big)\big)^\sharp\big\rangle \\
&\quad = *\big(\mathbf{d}H \wedge \big(\mathbf{d}C \wedge *(\mathbf{d}H \wedge \mathbf{d}C)\big)\big) \geq 0 \quad \text{(for } K \text{ even)}, \\
&\quad = - * \big(\mathbf{d}H \wedge \big(\mathbf{d}C \wedge *(\mathbf{d}H \wedge \mathbf{d}C)\big)\big) \leq 0 \quad \text{(for } K \text{ odd)}.
\end{aligned} \tag{11}$$

The distinction between the odd and even dimension cases is related to the fact that the operator $**$ acting on a codimension-one form in an even-dimensional Euclidean space returns the negative of that form (see Appendix A). We will assume that the sign of the dissipative vector field is always correspondingly adjusted to ensure that it dissipates H (and does not increase it).

In general, assuming M conserved quantities with $M < \dim(\mathbb{R}^K) - 1 = K - 1$, consider the vector field given by

$$-\big(*\big((\mathbf{d}C_1 \wedge \cdots \wedge \mathbf{d}C_M) \wedge *\big(\mathbf{d}H \wedge (\mathbf{d}C_1 \wedge \cdots \wedge \mathbf{d}C_M)\big)\big)\big)^\sharp, \tag{12}$$

where $C_l : \mathbb{R}^K \to \mathbb{R},\ l = 1, \ldots, M$.

Using (72) and wedge product properties again, we obtain

$$\begin{aligned}
&-\langle (\mathbf{d}C_l)^\sharp, \big(*\big((\mathbf{d}C_1\wedge\cdots\wedge\mathbf{d}C_M)\wedge *\big(\mathbf{d}H\wedge(\mathbf{d}C_1\wedge\cdots\wedge\mathbf{d}C_M)\big)\big)\big)^\sharp\rangle \\
&\quad = (-1)^K * \big(\big((\mathbf{d}C_l\wedge\mathbf{d}C_1\wedge\cdots\wedge\mathbf{d}C_l)\wedge(\mathbf{d}C_{l+1}\wedge\cdots\wedge\mathbf{d}C_M)\big) \\
&\qquad \wedge *\big(\mathbf{d}H\wedge(\mathbf{d}C_1\wedge\cdots\wedge\mathbf{d}C_M)\big)\big) \\
&\quad = 0
\end{aligned}\tag{13}$$

and, similarly, for K even,

$$\begin{aligned}
&-\langle (\mathbf{d}H)^\sharp, \big(*\big((\mathbf{d}C_1\wedge\cdots\wedge\mathbf{d}C_M)\wedge *\big(\mathbf{d}H\wedge(\mathbf{d}C_1\wedge\cdots\wedge\mathbf{d}C_M)\big)\big)\big)^\sharp\rangle \\
&\quad = *\big(\big(\mathbf{d}H\wedge\big(\mathbf{d}C_1\wedge\cdots\wedge\mathbf{d}C_M\big)\big)\wedge *\big(\mathbf{d}H\wedge(\mathbf{d}C_1\wedge\cdots\wedge\mathbf{d}C_M)\big)\big) \\
&\quad \geq 0
\end{aligned}\tag{14}$$

and, for K odd,

$$-\langle (\mathbf{d}H)^\sharp, \big(*\big((\mathbf{d}C_1\wedge\cdots\wedge\mathbf{d}C_M)\wedge *\big(\mathbf{d}H\wedge(\mathbf{d}C_1\wedge\cdots\wedge\mathbf{d}C_M)\big)\big)\big)^\sharp\rangle \leq 0. \tag{15}$$

Note that the construction above being purely geometric, the conserved functions need not even arise from symmetries, they could be, for example, holonomic constraints or Casimir functions.

We are now ready to state our first proposition.

Proposition 2.1 *Given a smooth Hamiltonian vector field X_H on $\mathbb{R}^K$ with $M < K-1$ conserved quantities or, more generally, a smooth vector field X with $M+1$ conserved quantities $H, C_1, \ldots, C_M : \mathbb{R}^K \to \mathbb{R}$, a dissipative total vector field X_D of the form* (6) *obtained by defining*

$$X_d = (-1)^K \big(*\big(\big(\mathbf{d}C_1\wedge\cdots\wedge\mathbf{d}C_M\big)\wedge *\big(\mathbf{d}H\wedge(\mathbf{d}C_1\wedge\cdots\wedge\mathbf{d}C_M)\big)\big)\big)^\sharp \tag{16}$$

has the property of dissipating the function H while conserving the level sets of $C_1, \ldots, C_M$.

We note that X_d as defined above becomes zero at:

1. Critical points of one or more of $H, C_1, \ldots, C_M$
2. Points where $(\mathbf{d}H, \mathbf{d}C_1, \ldots, \mathbf{d}C_M)$ fails to be a linearly independent set.

Moreover, X_d cannot be constructed if the intersection of the level sets of $C_1, \ldots, C_M$ degenerates to a curve, which happens if, for example, $K = M - 1$. Since X_H is tangent to this curve and to the level set of H, it follows that X_d, in this case, is also tangent to the level set of H. To construct a dissipative field, one would therefore have to relax the constraint of conserving *all* the C_ls. This feature is seen in the phase space of the two-point-vortex problem.

Matrix Formulation We now present an equivalent matrix formulation of the above ideas and state two important properties of X_d.

By its definition, (16) , X_d is linear in $\mathbf{d}H$. It follows that it is expressible as

$$X_d := A \cdot \nabla H, \tag{17}$$

where A is a $K \times K$ matrix. Moreover, we make the following association which will be useful later on. For any vector $v \in \mathbb{R}^K$,

$$A \cdot v \equiv (-1)^K \big(*\big((\mathbf{d}C_1 \wedge \cdots \wedge \mathbf{d}C_M) \wedge *\big(v^\flat \wedge (\mathbf{d}C_1 \wedge \cdots \wedge \mathbf{d}C_M)\big)\big)\big)^\sharp. \tag{18}$$

We first show that the matrix A is symmetric. To see this note that since any row a_r of A satisfies

$$\langle X_d, e_r \rangle = \langle a_r, \nabla H \rangle, \quad r = 1, \ldots, K,$$

where e_r is the standard unit vector with 1 in the rth slot and 0 everywhere else, we obtain from (16)

$$\begin{aligned} \langle X_d, e_r \rangle &= (-1)^{2K-1} * \big(e_r^\flat \wedge \big((\mathbf{d}C_1 \wedge \cdots \wedge \mathbf{d}C_M) \\ &\quad \wedge *\big(\mathbf{d}H \wedge (\mathbf{d}C_1 \wedge \cdots \wedge \mathbf{d}C_M)\big)\big)\big) \\ &= (-1)^{2K-1} * \big(\mathbf{d}H \wedge \big((\mathbf{d}C_1 \wedge \cdots \wedge \mathbf{d}C_M) \\ &\quad \wedge *\big(e_r^\flat \wedge (\mathbf{d}C_1 \wedge \cdots \wedge \mathbf{d}C_M)\big)\big)\big), \end{aligned} \tag{19}$$

and so a_r is given by the vector field

$$(-1)^K \big(*\big((\mathbf{d}C_1 \wedge \cdots \wedge \mathbf{d}C_M) \wedge *\big(e_r^\flat \wedge (\mathbf{d}C_1 \wedge \cdots \wedge \mathbf{d}C_M)\big)\big)\big)^\sharp. \tag{20}$$

But comparing the above with (18), we see that it is the same as $A \cdot e_r$, which is the rth column of A. Hence A is symmetric.

Next, since

$$A \cdot \nabla C_l = 0, \quad l = 1, \ldots, M \tag{21}$$

(which can be directly seen by replacing $\mathbf{d}H$ by $\mathbf{d}C_l$ in (16)), it follows that A has nonzero kernel and is singular.

Rate of Dissipation The time rate of dissipation of H by X_D is given by

$$\frac{dH}{dt} = \langle \nabla H, X_D \rangle = \langle \nabla H, X_H + X_d \rangle = \langle \nabla H, X_d \rangle. \tag{22}$$

For any X_d, we note that the vector field ϕX_d, where $\phi : \mathbb{R}^K \times \mathbb{R}^l \to \mathbb{R}^+$ with $\mathbb{R}^l$ some parameter space, also conserves the level sets of $C_1, \ldots, C_M$ and dissipates H. It follows that if the rate of dissipation is some *prescribed* function $-\phi_d$, where $\phi_d : \mathbb{R}^K \to \mathbb{R}^+$, then a straightforward normalization and redefinition of X_d as

$$\bar{X}_d = \frac{\phi_d X_d}{|\langle \nabla H, X_d \rangle|} \tag{23}$$

gives us a dissipative field with the same two features of X_d in addition to dissipating H at exactly the rate $-\phi_d$. However, obviously, $\bar{X}_d$ has singular points and smoothing $\bar{X}_d$ in the neighborhood of such points may lead to the dissipation rate being satisfied only approximately in such neighborhoods. It should be noted that $\bar{X}_d$ has the correct units of "phase space coordinate per time".

Higher-order Dissipative Fields We now present an important generalization of the vector fields X_d.

Proposition 2.2 *Consider the kth order dissipative vector field* ($k \in \mathbb{Z}^+$)

$$X_d^k := (-1)^{k-1}(A)^k \cdot \nabla H \equiv (-1)^{k-1} \overbrace{A \cdot A \cdot \cdots \cdot A}^{k \text{ times}} \cdot \nabla H. \tag{24}$$

For any k, X_d^k *has the same properties as* X_d, *i.e., it dissipates* H *while conserving* $C_1, \ldots, C_M$.

Proof of Proposition 2.2 First,

$$\begin{aligned} \langle \nabla C_l, X_d^k \rangle &= (-1)^{k-1} \langle \nabla C_l, A \cdot (A)^{k-1} \cdot \nabla H \rangle \\ &= (-1)^{k-1} \langle A \cdot \nabla C_l, (A)^{k-1} \cdot \nabla H \rangle \quad [A \text{ symmetric}] \\ &= 0 \quad [\text{from } (21)]. \end{aligned} \tag{25}$$

Next, if k is even,

$$\begin{aligned} \langle \nabla H, X_d^k \rangle &= -\langle \nabla H, (A)^k \cdot \nabla H \rangle \\ &= -\langle A \cdot \nabla H, (A)^{k-1} \cdot \nabla H \rangle \quad [A \text{ symmetric}]. \end{aligned} \tag{26}$$

Repeating this process $k/2 - 1$ times gives

$$\langle \nabla H, X_d^k \rangle = -\langle (A)^{k/2} \cdot \nabla H, (A)^{k/2} \cdot \nabla H \rangle \le 0. \tag{27}$$

Next, if k is odd,

$$\begin{aligned} \langle \nabla H, X_d^k \rangle &= \langle \nabla H, (A)^k \cdot \nabla H \rangle \\ &= \langle A \cdot \nabla H, (A)^{k-1} \cdot \nabla H \rangle \quad [A \text{ symmetric}]. \end{aligned} \tag{28}$$

Repeating this process $(k-1)/2 - 1$ times gives

$$\langle \nabla H, X_d^k \rangle = \langle (A)^{(k-1)/2} \cdot \nabla H, (A)^{(k+1)/2} \cdot \nabla H \rangle. \tag{29}$$

But from (18), introducing the notation $\lambda = \mathbf{d}C_1 \wedge \cdots \wedge \mathbf{d}C_M$, we have

$$\begin{aligned} &\langle (A)^{(k-1)/2} \cdot \nabla H, (A)^{(k+1)/2} \cdot \nabla H \rangle \\ &\quad = \langle (A)^{(k-1)/2} \cdot \nabla H, (-1)^K \left(*\left(\lambda \wedge *\left(\left((A)^{(k-1)/2} \cdot \nabla H \right)^\flat \wedge \lambda \right) \right) \right)^\sharp \rangle \end{aligned}$$

$$= (-1)^{2K-1}\big((A)^{(k-1)/2}\cdot\nabla H\big)^{\flat}\wedge\Big(\lambda\wedge *\Big(\big((A)^{(k-1)/2}\cdot\nabla H\big)^{\flat}\wedge\lambda\Big)\Big)$$
$$\leq 0. \tag{30}$$

□

The X_d^ks allow for an interesting generalization of (6) and Proposition 2.1 as well. As for any X_d, we have that for any X_d^k, the vector field $\phi_k X_d^k$, where $\phi_k:\mathbb{R}^K\times\mathbb{R}^l\to\mathbb{R}^+$ with $\mathbb{R}^l$ some parameter space, also conserves the level sets of $C_1,\ldots,C_M$ and dissipates H. Therefore, we can state the following.

Proposition 2.3 *Given a smooth Hamiltonian vector field X_H on $\mathbb{R}^K$ with $M<K-1$ conserved quantities or, more generally, a smooth vector field X with $M+1$ conserved quantities $H, C_1,\ldots,C_M:\mathbb{R}^K\to\mathbb{R}$, a dissipative total vector field X_D of the form*

$$X_D = X_H + \mathfrak{X}_d \tag{31}$$

obtained by defining

$$\mathfrak{X}_d = \sum_{k=1}^{l}\phi_k X_d^k, \tag{32}$$

where $l\in\mathbb{Z}^+$, has the property of dissipating the function H while conserving the level sets of $C_1,\ldots,C_M$.

A consequence of the above proposition is that the dissipative field $\mathfrak{X}_d$, for appropriate choices of ϕ_k and depending on the eigenvalues of A, can be written as an infinite series that converges to a function of A. For example, with $\phi_k := \psi^k/k!$, for some $\psi:\mathbb{R}^K\times\mathbb{R}^l\to\mathbb{R}^+$, one obtains $\mathfrak{X}_d = (\mathrm{Id} - e^{-\psi A})\cdot\nabla H$. As a particular case, ψ can be chosen as a (purely) time-dependent function, for example, $\psi := t$, which gives $\mathfrak{X}_d = (\mathrm{Id} - e^{-At})\cdot\nabla H$. For functions other than the exponential, the convergence of the series may place a restriction on the domain of definition of $\mathfrak{X}_d$.

3 Relation to Nambu–Poisson Brackets

Before proceeding to apply the above methodology to point vortex models, we briefly describe its relation to Nambu–Poisson brackets.[2]

The Nambu bracket was first introduced in Nambu (1973). A recent discussion may be found in, for example, Holm (2008a, 2008b). Nambu proposed the following bracket on $\mathbb{R}^3$:

$$\{F,H,C\} = -\nabla C.\nabla F\times\nabla H,\qquad F,H,C:\mathbb{R}^3\to\mathbb{R}, \tag{33}$$

[2] I am grateful to Darryl D. Holm and an anonymous referee for suggesting this relation.

which, as in the case of Poisson brackets, can be associated with a smooth vector field

$$X_{H,C} = \nabla C \times \nabla H \equiv \{\mathbf{x}, H\}, \quad \mathbf{x} \in \mathbb{R}^3, \tag{34}$$

along which F changes according to

$$\frac{dF}{dt} = \{F, H, C\}. \tag{35}$$

In fact, (33) is also a Poisson bracket—termed a Nambu–Poisson bracket—parameterized by the function C, see, for example, (Holm 2008b; Takhtajan 1994),

$$\{F, H\}_C = \{F, H, C\}, \tag{36}$$

and so $X_{H,C}$ is Hamiltonian. From (34) we note that $X_{H,C}$ is tangent to the level sets of C and H.

Equip $\mathbb{R}^3$ with the standard Euclidean metric. Define now a *double Nambu–Poisson bracket* associated with (36) following the general definition of a double bracket on a Poisson manifold equipped with a metric (Bloch et al. 1996),

$$\{\{F, H\}\}_C(\mathbf{x}) := \langle X_{F,C}(\mathbf{x}), X_{H,C}(\mathbf{x})\rangle. \tag{37}$$

One then easily checks that the vector field along which a function F changes according to

$$\frac{dF}{dt} = -\{\{F, H\}\}_C \tag{38}$$

is the dissipative vector field (7).

This relation can be generalized to higher-dimensional Euclidean spaces. Recall the generalization of (35) that Nambu proposed for $\mathbb{R}^K$ (Nambu 1973),

$$\begin{aligned} \frac{dF}{dt} &= \frac{\partial(F, H_1, \ldots, H_{K-1})}{\partial(x_1, \ldots, x_K)}, \quad F, H_1, \ldots, H_{K-1} : \mathbb{R}^K \to \mathbb{R}, \\ &=: \{F, H_1, \ldots, H_{K-1}\}. \end{aligned} \tag{39}$$

Noting that the right-hand side of (39) is $\mathbf{d}F \wedge \mathbf{d}H_1 \wedge \cdots \wedge \mathbf{d}H_{K-1}(e_1, \ldots, e_K)$, rewrite the equation in terms of exterior forms as

$$\begin{aligned} *\big(\mathbf{d}F \wedge *X^\flat_{H_1,\ldots,H_{K-1}}\big) &= \mathbf{d}F \wedge \mathbf{d}H_1 \wedge \cdots \wedge \mathbf{d}H_{K-1}(e_1, \ldots, e_K) \\ &= *(\mathbf{d}F \wedge \mathbf{d}H_1 \wedge \cdots \wedge \mathbf{d}H_{K-1}), \end{aligned} \tag{40}$$

the last line following from (66). This implies

$$X^\flat_{H_1,\ldots,H_{K-1}} = (-1)^{K-1} * (\mathbf{d}H_1 \wedge \cdots \wedge \mathbf{d}H_{K-1}), \tag{41}$$

where the vector field $X_{H_1,\ldots,H_{K-1}}$ is Hamiltonian relative to the Nambu–Poisson bracket

$$\{F, H_1\}_{H_2,\ldots,H_{K-1}} := \{F, H_1, \ldots, H_{K-1}\}. \tag{42}$$

The reader is referred to Takhtajan (1994) for an explanation of how the above bracket is Poisson. As in the $\mathbb{R}^3$ case defining the double Nambu–Poisson bracket as

$$\{\{F, H_1\}\}_{H_2,\ldots,H_{K-1}} := \left\langle X_{F,H_2,\ldots,H_{K-1}}(\mathbf{x}), X_{H_1,\ldots,H_{K-1}}(\mathbf{x})\right\rangle, \tag{43}$$

it follows that the equation

$$\frac{dF}{dt} = -\{\{F, H_1\}\}_{H_2,\ldots,H_{K-1}} \tag{44}$$

gives the rate of change of F along a vector field which, after some exterior algebra, is the dissipative vector field (16).

We summarize the results in this section with the following:

Proposition 3.1 *For the case $M = K - 2$, the first-order dissipative vector field* (16) *is the vector field associated with the double Nambu–Poisson bracket* (43).

The restriction $M = K - 2$ is obviously related to the fact that a Nambu system on a K-dimensional phase space requires $K - 1$ Hamiltonians for its formulation.

Before concluding this section, we mention a few references related to the discussion here. Makhaldiani (1998) has shown that the Hamiltonian three-point-vortex problem in symmetry reduced coordinates has a Nambu bracket formulation. The Nambu formulation for the incompressible Euler equations is presented in Nevir and Blender (1993). More on this Nambu formulation and that for other nonlinear evolution equations can be found in the papers by Guha (2001, 2004).

4 Dissipative N-point-vortex Models

With the theory of the previous sections in place, we now present the dissipative N-point-vortex models.

Corresponding to Proposition 2.1, we have:

Proposition 4.1 *Given the Hamiltonian N-point-vortex vector field* (1) *on $P \equiv \mathbb{R}^{2N}\backslash\triangle$ with the conserved quantities* (4) *associated with the* SE(2) *symmetries, a dissipative N-point-vortex vector field X_{DPV} of the form*

$$X_{\mathrm{DPV}} = X_{\mathrm{HPV}} + X_{\mathrm{dpv}} \tag{45}$$

obtained by defining

$$X_{\mathrm{dpv}} = \Big(*\big((\mathbf{d}I_1 \wedge \mathbf{d}I_2 \wedge \mathbf{d}I_3) \wedge *\big(\mathbf{d}H_{\mathrm{pv}} \wedge (\mathbf{d}I_1 \wedge \mathbf{d}I_2 \wedge \mathbf{d}I_3)\big)\big)\Big)^{\sharp} \tag{46}$$

has the property of dissipating the point vortex kinetic energy Hamiltonian (3) *while conserving the level sets of* (4).

The set $\triangle$ in the statement of the proposition is the set of all collision points of the vortices, $\triangle := \{(x_1, y_1, \dots, x_N, x_N) \mid (x_j, y_j) = (x_l, y_l),\ j \neq l\}$.

Example: Two-point-vortex Model As mentioned previously, in the two-point-vortex model ($N = 2$) the problem of constructing X_{dpv} (or X_{DPV}) is over-constrained. Nevertheless, to start off with a simple example, we lessen these constraints and present an X_{dpv} which is required to preserve, say, only one of the invariants (4),

$$I_1 = \Gamma_1 x_1 + \Gamma_2 x_2.$$

With choice of volume form $dx_1 \wedge dy_1 \wedge dx_2 \wedge dy_2$, following the method outlined in the previous section, we can then show that X_{DPV} takes the form

$$\begin{aligned}\frac{dx_1}{dt} &= \frac{1}{\Gamma_1}\frac{\partial H_{\text{pv}}}{\partial y_1}\\ &\quad - c\left(\{H_{\text{pv}}, I_1\}_{x_1,x_2}\frac{\partial I_1}{\partial x_2} + \{H_{\text{pv}}, I_1\}_{x_1,y_1}\frac{\partial I_1}{\partial y_1} + \{H_{\text{pv}}, I_1\}_{x_1,y_2}\frac{\partial I_1}{\partial y_2}\right), \quad (47)\\ \frac{dy_1}{dt} &= -\frac{1}{\Gamma_1}\frac{\partial H_{\text{pv}}}{\partial x_1}\\ &\quad - c\left(\{H_{\text{pv}}, I_1\}_{y_1,x_1}\frac{\partial I_1}{\partial x_1} + \{H_{\text{pv}}, I_1\}_{y_1,x_2}\frac{\partial I_1}{\partial x_2} + \{H_{\text{pv}}, I_1\}_{y_1,y_2}\frac{\partial I_1}{\partial y_2}\right), \quad (48)\\ \frac{dx_2}{dt} &= \frac{1}{\Gamma_2}\frac{\partial H_{\text{pv}}}{\partial y_2}\\ &\quad - c\left(\{H_{\text{pv}}, I_1\}_{x_2,y_1}\frac{\partial I_1}{\partial y_1} + \{H_{\text{pv}}, I_1\}_{x_2,x_1}\frac{\partial I_1}{\partial x_1} + \{H_{\text{pv}}, I_1\}_{x_2,y_2}\frac{\partial I_1}{\partial y_2}\right), \quad (49)\\ \frac{dy_2}{dt} &= -\frac{1}{\Gamma_2}\frac{\partial H_{\text{pv}}}{\partial x_2}\\ &\quad - c\left(\{H_{\text{pv}}, I_1\}_{y_2,y_1}\frac{\partial I_1}{\partial y_1} + \{H_{\text{pv}}, I_1\}_{y_2,x_1}\frac{\partial I_1}{\partial x_1} + \{H_{\text{pv}}, I_1\}_{y_2,x_2}\frac{\partial I_1}{\partial x_2}\right). \quad (50)\end{aligned}$$

The details of the computation of X_{dpv} are presented in Appendix B. In the above,

$$\{H_{\text{pv}}, I_1\}_{x_i,y_j} = \frac{\partial H_{\text{pv}}}{\partial x_i}\frac{\partial I_1}{\partial y_j} - \frac{\partial H_{\text{pv}}}{\partial y_j}\frac{\partial I_1}{\partial x_i}, \quad (51)$$

and c is a constant needed to make the equation homogeneous with regard to units. Alternatively, if the rate of dissipation $-\phi_d$ is prescribed, then X_{dpv} can be normalized to $\bar{X}_{\text{dpv}}$ as in (23) and c is then the positive function defined as

$$c := \frac{\phi_d}{|\langle \nabla H, X_{\text{dpv}}\rangle|}. \quad (52)$$

Expanding the terms in X_{dpv}, we obtain

$$\frac{dx_1}{dt} = \frac{1}{\Gamma_1}\frac{\partial H_{\text{pv}}}{\partial y_1} - c\left(\Gamma_2\left(\Gamma_2\frac{\partial H_{\text{pv}}}{\partial x_1} - \Gamma_1\frac{\partial H_{\text{pv}}}{\partial x_2}\right)\right), \tag{53}$$

$$\frac{dy_1}{dt} = -\frac{1}{\Gamma_1}\frac{\partial H_{\text{pv}}}{\partial x_1} - c\left(\left(\Gamma_1^2 + \Gamma_2^2\right)\frac{\partial H_{\text{pv}}}{\partial y_1}\right), \tag{54}$$

$$\frac{dx_2}{dt} = \frac{1}{\Gamma_2}\frac{\partial H_{\text{pv}}}{\partial y_2} - c\left(\Gamma_1\left(\Gamma_1\frac{\partial H_{\text{pv}}}{\partial x_2} - \Gamma_2\frac{\partial H_{\text{pv}}}{\partial x_1}\right)\right), \tag{55}$$

$$\frac{dy_2}{dt} = -\frac{1}{\Gamma_2}\frac{\partial H_{\text{pv}}}{\partial x_2} - c\left(\left(\Gamma_1^2 + \Gamma_2^2\right)\frac{\partial H_{\text{pv}}}{\partial y_2}\right). \tag{56}$$

In this simple example, the symmetry and dissipative properties can be directly checked. Indeed,

$$\langle \nabla I_1, X_{\text{dpv}} \rangle = 0,$$

as can be easily verified, and

$$\begin{aligned}
\langle \nabla H_{\text{pv}}, X_{\text{dpv}} \rangle = -c\Bigg(&\Gamma_2^2\left(\frac{\partial H_{\text{pv}}}{\partial x_1}\right)^2 + \left(\Gamma_1^2 + \Gamma_2^2\right)\left(\left(\frac{\partial H_{\text{pv}}}{\partial y_1}\right)^2 + \left(\frac{\partial H_{\text{pv}}}{\partial y_2}\right)^2\right) \\
&+ \Gamma_1^2\left(\frac{\partial H_{\text{pv}}}{\partial x_2}\right)^2 - 2\Gamma_1\Gamma_2\frac{\partial H_{\text{pv}}}{\partial x_1}\frac{\partial H_{\text{pv}}}{\partial x_2}\Bigg) \\
= -c\Bigg(&\left(\Gamma_1\frac{\partial H_{\text{pv}}}{\partial x_2} - \Gamma_2\frac{\partial H_{\text{pv}}}{\partial x_1}\right)^2 + \left(\Gamma_1^2 + \Gamma_2^2\right)\left(\left(\frac{\partial H_{\text{pv}}}{\partial y_1}\right)^2 \right.\\
&\left. + \left(\frac{\partial H_{\text{pv}}}{\partial y_2}\right)^2\right)\Bigg) \\
\leq 0. &
\end{aligned} \tag{57}$$

Example: Three-point-vortex Model In the three-point vortex model, the invariants (4) take the form

$$I_1 = \Gamma_1 x_1 + \Gamma_2 x_2 + \Gamma_3 x_3, \tag{58}$$

$$I_2 = \Gamma_1 y_1 + \Gamma_2 y_2 + \Gamma_3 y_3, \tag{59}$$

$$I_3 = \Gamma_1\left(\frac{x_1^2 + y_1^2}{2}\right) + \Gamma_2\left(\frac{x_2^2 + y_2^2}{2}\right) + \Gamma_3\left(\frac{x_3^2 + y_3^2}{2}\right). \tag{60}$$

The elements of the matrix A for this problem are presented in Appendix C. The first row is computed as per (20) and similarly the other rows. As in the two-point-vortex model, X_{dpv} has to be multiplied by a constant to homogenize the equations with regard to units or, for prescribed dissipation rates, $\bar{X}_{\text{dpv}}$ can be used.

In Figs. 1, 2, and 3, we present an example of dissipative point vortex dynamics for the simplest case of $N = 2$. To circumvent the difficulty of constructing the model

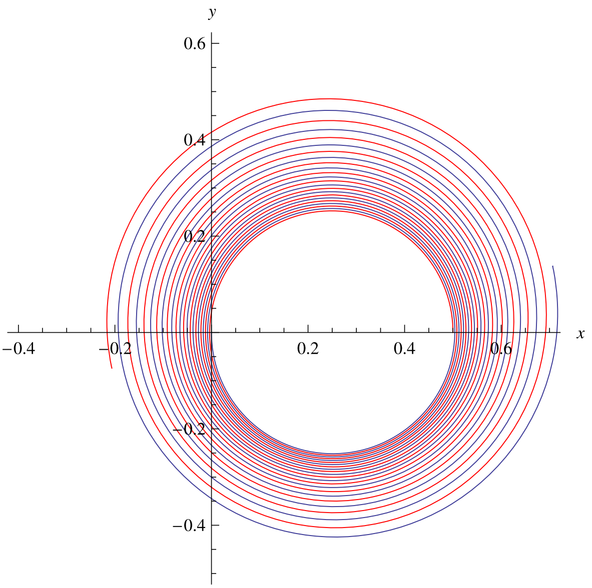

Fig. 1 Dissipative two point vortex trajectories in a three-point-vortex model in which the third vortex (not shown in the figure) is sufficiently far away, $\Gamma_1 = \Gamma_2 = 1$, $\Gamma_3 = -2$, $x_1(0) = y_1(0) = 0$, $x_2(0) = 0.5$, $y_2(0) = 0$, $x_3(0) = 1000$, $y_3(0) = 1$. Prescribed dissipation rate of Hamiltonian function $= 0.001$ and time of evolution $= 100$

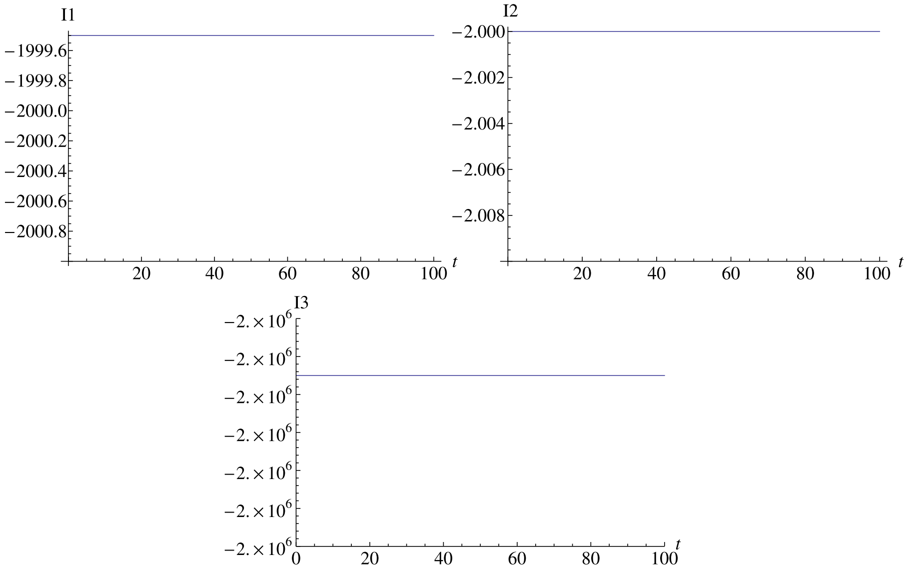

Fig. 2 Preservation of invariants I_1, I_2, and I_3

in this case (as discussed above), we focus on two-point-vortex dynamics within a three-point-vortex configuration in which the third vortex is sufficiently far away from the vortex pair to have negligible influence on its dynamics. The sum of the vortex strengths is zero, and the rate of dissipation of the Hamiltonian function is

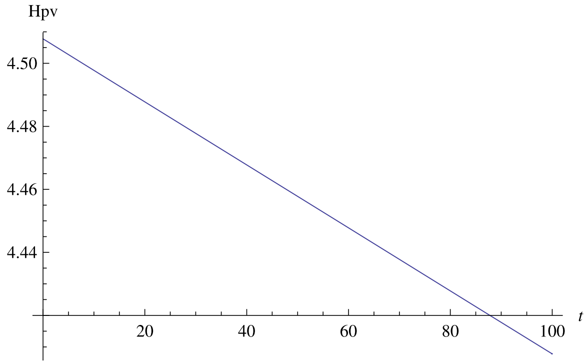

Fig. 3 Dissipation of Hamiltonian function H_{pv}

fixed at an arbitrary constant value. An outward spiraling of the vortex trajectories—suggestive of the effects of Navier–Stokes vorticity diffusion—is seen in Fig. 1. As a note of comparison, the Hamiltonian trajectories (not shown) of the two point vortices starting from the same configuration would almost coincide with the circle centered at $(0.25, 0)$ and radius 0.25.[3]

5 Summary, Discussions, and Future Directions

In this paper, we have presented a method for constructing finite-dimensional dissipative vector fields which have the property of conserving the level sets of a prescribed number of functions while dissipating another—typically, an energy function. The motivating application is the construction of a dissipative N-point-vortex model in the plane. The underlying idea comes from elementary geometry in $\mathbb{R}^3$, and the extension to higher-dimensional spaces is done using exterior algebra—in particular, through the use of wedge products and Hodge star operators. A remarkable feature of this method is that an infinite number of such dissipative vector fields can be constructed by repeated application of a symmetric linear operator (matrix). The construction of this matrix can get algebraically cumbersome as the dimension of the system increases. However, we would not anticipate any major computational challenges in the implementation of the method unless the system dimension is really large.

We should emphasize again the conceptual simplicity of this method. The formal constructs of geometric mechanics, used in the method of Bloch et al. (1996), for example, such as cotangent bundle, Euler–Poincaré formulation and coadjoint orbits are not needed here. Some familiarity with exterior algebra is however needed. A relation to double bracket dissipation via Nambu–Poisson brackets exists. This implies that the methodology in this paper could be applied to any finite-dimensional dynamical system with a Nambu formulation. Examples of several such systems may be found in the papers by Chatterjee (1996) and Baleanu and Makhaldiani (1999).

[3] In the absence of the third vortex, this circle would be the exact trajectory of both vortices.

Another possible similarity with double bracket dissipation—specifically the idea of a normal metric detailed in Bloch et al.—may also exist. The symmetric matrix A is also, obviously, a symmetric tensor, and one may speculate that it is the tensor of a metric defined on the intersecting level sets.

From a point vortex perspective, an important question is: in what sense does the model dynamics, in its present form, represent the Navier–Stokes dynamics of coherent vortical structures? Since this model does not explicitly take into account core dynamics, the answer to the question cannot be strongly in the affirmative. A more relevant question to ask therefore is: do these models capture Navier–Stokes behavior better than the Hamiltonian N-point-vortex model? One may speculate that this model predicts the movements of coherent vortices better than the Hamiltonian model as long as the vortices remain coherent and sufficiently separated. Vortical events dominated by core dynamics, such as merging and filamentation, may still be missed by this model. However, it should be pointed out that the model can be developed further. The utility of the higher-order dissipative vector fields remains to be explored; other possible avenues for improvements are: (i) using point vortex strengths Γ_i that are no longer fixed but continuously changing in time, (ii) incorporating the decay laws for higher-order moments of vorticity distributions of the incompressible Navier–Stokes equations (Ting and Klein 1991)—the methodology proposed in this paper is easily extended to the case where the constructed vector field dissipates $L(> 1)$ functions while conserving M others by simply constructing L affine dissipative vector fields each of which dissipates one of the L functions while conserving the remaining $M + L - 1$, and (iii) incorporating ideas from the vortex blob methods of Chorin or the Hamiltonian vortex patch models described in Melander et al. (1986, 1988); see also (Fliert and Groesen 1992). These issues will be explored more in the future.

Some other directions of future research also present themselves. The Hamiltonian N-point-vortex model has also been studied on other topological surfaces (Boatto and Koiller 2008) and, in particular, on the sphere (Kidambi and Newton 1998). Since our methodology uses essentially "coordinate-free" objects, it should be applicable to Hamiltonian N-point-vortex models on these surfaces as well. In the same vein, we also anticipate applications to partial differential evolution equations. Finally, the affine manner in which the dissipative vector field is introduced implies that the method is in principle applicable to control systems as well.

Acknowledgements This paper owes its existence to a number of people besides the author. First and foremost, I would like to thank Darryl D. Holm for suggesting the problem of constructing dissipative singular vortex models. Darryl graciously hosted me for two months at Imperial College, London, where most of this work was carried out. His patient listening and insightful comments are greatly appreciated. I would also like to thank Darren G. Crowdy for co-hosting me at Imperial. Part of the work was carried out at the Aerodynamisches Institut of RWTH-Aachen, Germany, under the auspices of an Alexander von Humboldt Summer Research Fellowship. Wolfgang Schröder and all the members of the institute deserve a thanks for their hospitality during my stay. The three anonymous referees and the responsible editor Anthony Bloch provided several useful comments and suggestions, which helped to improve the exposition of the paper. Last, but not least, I would like to thank Steve Alpern for helping me cope with London living.

Appendix A: Exterior Algebra Basics

Good and thorough introductions to exterior algebra may be found, for example, in the texts by Abraham et al. (1988) (Chap. 6) and Carmo (1994). Here we present only a few basic aspects, borrowed from those texts, relevant to this paper.

Let $\Omega^k(M)$ denote the space of k-forms on an n-dimensional manifold M ($k = 0, 1, \dots, n$). If $\alpha \in \Omega^1(M)$, then, at each $m \in M$, $\alpha(m)$ can be identified with an element of T_m^*M, the dual space of $T_m M$.

A Riemannian metric g on M defines an isomorphism between $T_m M$ and $T_m^* M$ at each $m \in M$. This extends to a vector bundle isomorphism between TM and T^*M, i.e., between vector fields and 1-forms on M defined by $\alpha(m)() = \langle\, , X(m)\rangle_m$, where $\alpha \in \Omega^1(M)$, X is a vector field, i.e., a section of the vector bundle $\pi_t : TM \to M$, and $\langle\, ,\rangle_m$ is the inner product at $m \in M$. The musical symbol$^\flat$ denotes the map from vector fields to 1-forms,

$$X^\flat(m)() = \big\langle\, , X(m)\big\rangle_m, \tag{61}$$

and the musical symbol $^\sharp$ denotes the inverse of $^\flat$ (1-forms to vector fields),

$$\alpha(m)() = \big\langle\, , \alpha^\sharp(m)\big\rangle_m. \tag{62}$$

For a smooth function $C : \mathbb{R}^n \to \mathbb{R}$, $\mathbf{d}C$ is a 1-form, and

$$\mathbf{d}C \equiv (\nabla C)^\flat. \tag{63}$$

If $(e_1, \dots, e_n)$ represents an orthonormal basis of $T_m M$, then there exists a unique orthonormal basis of $T_m^* M$, $(e^1, \dots, e^n)$ satisfying $e^i(e_j) = 0,\ j \neq i$, and $e^i(e_i) = 1$. Note that if $M = \mathbb{R}^n$, then the bases are independent of the point m. Moreover, if $x_i : \mathbb{R}^n \to \mathbb{R},\ i = 1, \dots, n$, are coordinates, then $(dx_1, \dots, dx_n)$ represents an orthonormal basis for $(\mathbb{R}^n)^*$. More generally,

$$\{dx_{i_1} \wedge \cdots \wedge dx_{i_k} \mid 1 \le i_1 < i_2 < \cdots < i_k \le n\} \tag{64}$$

represents a basis for $\Omega^k(\mathbb{R}^n)$.

Equip $\mathbb{R}^n$ with the standard Euclidean metric. Let $\gamma = \sum_{i=1}^n a_i dx_i$, $a_i : \mathbb{R}^n \to \mathbb{R}$, be a 1-form, and let $X : \mathbb{R}^n \to T\mathbb{R}^n \equiv \mathbb{R}^n \times \mathbb{R}^n$ be a vector field with $X \equiv (u_1, \dots, u_n)$, $u_i : \mathbb{R}^n \to \mathbb{R}$. It follows from (62) that

$$\gamma^\sharp = (a_1, \dots, a_n) \tag{65}$$

and from (61) that $X^\flat = \sum_{i=1}^n u_i\, dx_i$.

The Hodge star operator $*$ takes an element of $\Omega^k(M)$ to an element of $\Omega^{n-k}(M)$. It is defined as

$$(*\alpha)(e_{k+1}, \dots, e_n) = \alpha(e_1, \dots, e_k). \tag{66}$$

If $M = \mathbb{R}^n$ equipped with the standard Euclidean metric, then the Hodge star operator satisfies the following relations:

$$\alpha \wedge *\gamma = \gamma \wedge *\alpha, \tag{67}$$

$$*\mu = 1, \tag{68}$$

$$**\alpha = (-1)^{k(n-k)}\alpha, \tag{69}$$

where μ is the volume form. In particular, its action on the basis elements of $\Omega^k(\mathbb{R}^n)$ is given by

$$\begin{aligned}
&*(dx_{i_1} \wedge \cdots \wedge dx_{i_k}) = (-1)^\sigma (dx_{j_1} \wedge \cdots \wedge dx_{j_{n-k}}), \\
&\quad i_1 < \cdots < i_k,\ j_1 < \cdots < j_{n-k},
\end{aligned} \tag{70}$$

where $(i_1, \ldots, i_k, j_1, \ldots, j_{n-k})$ is a permutation of $(1, \ldots, n)$, and σ is 0 or 1 depending on whether the permutation is even or odd, respectively. For example, in $\mathbb{R}^4$, $*(dx_1 \wedge dx_3) = -dx_2 \wedge dx_4$, $*(dx_2 \wedge dx_3) = dx_1 \wedge dx_4$.

Now let $\alpha, \beta \in \Omega^1(\mathbb{R}^n)$, $\alpha = \sum_{i_l=1}^n a_{i_l}\, dx_{i_l}$, $\beta = \sum_{i_l=1}^n b_{i_l}\, dx_{i_l}$, $a_{i_l}, b_{i_l} : \mathbb{R}^n \to \mathbb{R}$, and let $\mu = dx_1 \wedge \cdots \wedge dx_n$ be the volume form. It follows that

$$\begin{aligned}
\alpha \wedge *\beta &= \left(\sum_{i_l=1}^n a_{i_l}\, dx_{i_l}\right) \wedge \left(\sum_{i_l=1}^n b_{i_l} * dx_{i_l}\right) \\
&= \left(\sum_{i_l=1}^n a_{i_l}\, dx_{i_l}\right) \wedge \left(\sum_{i_l=1}^n b_{i_l} (-1)^\sigma (dx_{j_1} \wedge \cdots \wedge dx_{j_{n-1}})\right),
\end{aligned} \tag{71}$$

where $(i_l, j_1, \ldots, j_{n-1})$ is a permutation of $(1, \ldots, n)$, and therefore,

$$\begin{aligned}
\alpha \wedge *\beta &= \sum_{i_l=1}^n a_{i_l} b_{i_l} (-1)^{2\sigma} \mu \\
&= \langle \alpha^\sharp, \beta^\sharp \rangle \mu \quad [\text{using (65)}] \\
\Rightarrow \quad *(\alpha \wedge *\beta) &= \langle \alpha^\sharp, \beta^\sharp \rangle \quad [\text{using (68)}].
\end{aligned} \tag{72}$$

The last equation is used at several points in the paper.

Appendix B: Computation of X_{dpv} for the two-point-vortex problem

We have

$$\mathbf{d}I_1 = \frac{\partial I_1}{\partial x_1} dx_1 + \frac{\partial I_1}{\partial y_1} dy_1 + \frac{\partial I_1}{\partial x_2} dx_2 + \frac{\partial I_1}{\partial y_2} dy_2 \tag{73}$$

and similarly $\mathbf{d}H_{\mathrm{pv}}$, so that

$$\begin{aligned}
*(\mathbf{d}H_{\mathrm{pv}} \wedge \mathbf{d}I_1) &= \left(\frac{\partial H_{\mathrm{pv}}}{\partial x_1}\frac{\partial I_1}{\partial y_1} - \frac{\partial H_{\mathrm{pv}}}{\partial y_1}\frac{\partial I_1}{\partial x_1}\right) dx_2 \wedge dy_2 \\
&\quad + \left(\frac{\partial H_{\mathrm{pv}}}{\partial x_1}\frac{\partial I_1}{\partial x_2} - \frac{\partial H_{\mathrm{pv}}}{\partial x_2}\frac{\partial I_1}{\partial x_1}\right) dy_2 \wedge dy_1
\end{aligned}$$

$$+\left(\frac{\partial H_{\mathrm{pv}}}{\partial x_1}\frac{\partial I_1}{\partial y_2}-\frac{\partial H_{\mathrm{pv}}}{\partial y_2}\frac{\partial I_1}{\partial x_1}\right)dy_1\wedge dx_2$$
$$+\left(\frac{\partial H_{\mathrm{pv}}}{\partial y_1}\frac{\partial I_1}{\partial x_2}-\frac{\partial H_{\mathrm{pv}}}{\partial x_2}\frac{\partial I_1}{\partial y_1}\right)dx_1\wedge dy_2$$
$$+\left(\frac{\partial H_{\mathrm{pv}}}{\partial y_1}\frac{\partial I_1}{\partial y_2}-\frac{\partial H_{\mathrm{pv}}}{\partial y_2}\frac{\partial I_1}{\partial y_1}\right)dx_2\wedge dx_1$$
$$+\left(\frac{\partial H_{\mathrm{pv}}}{\partial x_2}\frac{\partial I_1}{\partial y_2}-\frac{\partial H_{\mathrm{pv}}}{\partial y_2}\frac{\partial I_1}{\partial x_2}\right)dx_1\wedge dy_1, \tag{74}$$

and so

$$\begin{aligned}\mathbf{d}I_1\wedge *(\mathbf{d}H_{\mathrm{pv}}\wedge\mathbf{d}I_1)=&\Bigg[\Bigg(\{H_{\mathrm{pv}},I_1\}_{x_1,y_1}\frac{\partial I_1}{\partial x_1}-\{H_{\mathrm{pv}},I_1\}_{y_1,x_2}\frac{\partial I_1}{\partial x_2}\\&-\{H_{\mathrm{pv}},I_1\}_{y_1,y_2}\frac{\partial I_1}{\partial y_2}\Bigg)dx_1\wedge dx_2\wedge dy_2\Bigg]\\&+\Bigg[\Bigg(\{H_{\mathrm{pv}},I_1\}_{x_1,x_2}\frac{\partial I_1}{\partial x_2}+\{H_{\mathrm{pv}},I_1\}_{x_1,y_1}\frac{\partial I_1}{\partial y_1}\\&+\{H_{\mathrm{pv}},I_1\}_{x_1,y_2}\frac{\partial I_1}{\partial y_2}\Bigg)dx_2\wedge dy_2\wedge dy_1\Bigg]\\&+\Bigg[\Bigg(\{H_{\mathrm{pv}},I_1\}_{y_1,x_2}\frac{\partial I_1}{\partial y_1}+\{H_{\mathrm{pv}},I_1\}_{x_1,x_2}\frac{\partial I_1}{\partial x_1}\\&-\{H_{\mathrm{pv}},I_1\}_{x_2,y_2}\frac{\partial I_1}{\partial y_2}\Bigg)dy_1\wedge dx_1\wedge dy_2\Bigg]\\&+\Bigg[\Bigg(\{H_{\mathrm{pv}},I_1\}_{y_1,y_2}\frac{\partial I_1}{\partial y_1}+\{H_{\mathrm{pv}},I_1\}_{x_1,y_2}\frac{\partial I_1}{\partial x_1}\\&+\{H_{\mathrm{pv}},I_1\}_{x_2,y_2}\frac{\partial I_1}{\partial x_2}\Bigg)dy_1\wedge dx_2\wedge dx_1\Bigg],\end{aligned}\tag{75}$$

from which X_{dpv} is obtained as per its definition as

$$X_{\mathrm{dpv}}:=-\big(*\big(\mathbf{d}I_1\wedge *(\mathbf{d}H_{\mathrm{pv}}\wedge\mathbf{d}I_1)\big)\big)^{\sharp}. \tag{76}$$

Appendix C: Computation of the Matrix A for the Three-point-vortex Problem

The orthonormal basis for the dual of the point vortex phase space is chosen as $(dx_1, dy_1, dx_2, dy_2, dx_3, dy_3)$ and the volume form as $dx_1\wedge dy_1\wedge dx_2\wedge dy_2\wedge dx_3\wedge dy_3$. Using (20), the various rows of A are then computed. For example, the first row is given by

$$\left(*\left((\mathbf{d}I_1 \wedge \mathbf{d}I_2 \wedge \mathbf{d}I_3) \wedge *\left(dx_1 \wedge (\mathbf{d}I_1 \wedge \mathbf{d}I_2 \wedge \mathbf{d}I_3)\right)\right)\right)^\sharp,$$

the second row by

$$\left(*\left((\mathbf{d}I_1 \wedge \mathbf{d}I_2 \wedge \mathbf{d}I_3) \wedge *\left(dy_1 \wedge (\mathbf{d}I_1 \wedge \mathbf{d}I_2 \wedge \mathbf{d}I_3)\right)\right)\right)^\sharp,$$

and so on.

Below we present the various elements of the matrix A. The first row is given by

$$\begin{aligned} a_{11} &= -\Big(\big(\Gamma_1^2+\Gamma_2^2+\Gamma_3^2\big)\Gamma_2^2\Gamma_3^2(x_2-x_3)^2+\big(\Gamma_2^2+\Gamma_3^2\big)\big(\Gamma_1^2\Gamma_2^2(y_1-y_2)^2 \\ &\quad +\Gamma_1^2\Gamma_3^2(y_1-y_3)^2+\Gamma_2^2\Gamma_3^2(y_2-y_3)^2\big)\Big), && (77)\\ a_{12} &= \Gamma_1^2\big(\Gamma_2^2(x_1-x_2)+\Gamma_3^2(x_1-x_3)\big)\big(\Gamma_2^2(y_1-y_2)+\Gamma_3^2(y_1-y_3)\big), && (78)\\ a_{13} &= \Gamma_1\Gamma_2\Gamma_3^2(x_1-x_3)(x_2-x_3)\big(\Gamma_1^2+\Gamma_2^2+\Gamma_3^2\big) \\ &\quad +\Gamma_1\Gamma_2\big(\Gamma_1^2\Gamma_2^2(y_1-y_2)^2+\Gamma_1^2\Gamma_3^2(y_1-y_3)^2+\Gamma_2^2\Gamma_3^2(y_2-y_3)^2\big), && (79)\\ a_{14} &= \Gamma_1\Gamma_2\big(\Gamma_2^2(x_1-x_2)+\Gamma_3^2(x_1-x_3)\big)\big(\Gamma_1^2(-y_1+y_2)+\Gamma_3^2(y_2-y_3)\big), && (80)\\ a_{15} &= \Gamma_1\Gamma_3\Gamma_2^2(x_2-x_1)(x_2-x_3)\big(\Gamma_1^2+\Gamma_2^2+\Gamma_3^2\big) \\ &\quad +\Gamma_1\Gamma_3\big(\Gamma_1^2\Gamma_2^2(y_1-y_2)^2+\Gamma_1^2\Gamma_3^2(y_1-y_3)^2+\Gamma_2^2\Gamma_3^2(y_2-y_3)^2\big), && (81)\\ a_{16} &= -\Gamma_1\Gamma_3\big(\Gamma_2^2(x_1-x_2)+\Gamma_3^2(x_1-x_3)\big)\big(\Gamma_1^2(y_1-y_3)+\Gamma_2^2(y_2-y_3)\big); && (82) \end{aligned}$$

the second row by

$$\begin{aligned} a_{21} &= a_{12}, && (83)\\ a_{22} &= -\Gamma_2^2\Gamma_3^2\big(\Gamma_1^2+\Gamma_2^2+\Gamma_3^2\big)\big((y_2-y_3)^2\big)-\big(\Gamma_2^2+\Gamma_3^2\big)\big(\Gamma_2^2\Gamma_3^2(x_2-x_3)^2 \\ &\quad +\Gamma_1^2\Gamma_2^2(x_1-x_2)^2+\Gamma_1^2\Gamma_3^2(x_1-x_3)^2\big), && (84)\\ a_{23} &= -\Gamma_1\Gamma_2\big(\Gamma_1^2(x_1-x_2)+\Gamma_3^2(-x_2+x_3)\big)\big(\Gamma_2^2(y_1-y_2)+\Gamma_3^2(y_1-y_3)\big), && (85)\\ a_{24} &= \Gamma_1\Gamma_2\big(\Gamma_1^2\Gamma_2^2(x_1-x_2)^2+\Gamma_1^2\Gamma_3^2(x_1-x_3)^2+\Gamma_2^2\Gamma_3^2(x_2-x_3)^2\big) \\ &\quad +\Gamma_1\Gamma_2\Gamma_3^2\big(\Gamma_1^2+\Gamma_2^2+\Gamma_3^2\big)(y_1-y_3)(y_2-y_3), && (86)\\ a_{25} &= -\Gamma_1\Gamma_3\big(\Gamma_1^2(x_1-x_3)+\Gamma_2^2(x_2-x_3)\big)\big(\Gamma_2^2(y_1-y_2)+\Gamma_3^2(y_1-y_3)\big), && (87)\\ a_{26} &= \Gamma_1\Gamma_3\big(\Gamma_1^2\Gamma_2^2(x_1-x_2)^2+\Gamma_1^2\Gamma_3^2(x_1-x_3)^2+\Gamma_2^2\Gamma_3^2(x_2-x_3)^2\big) \\ &\quad +\Gamma_1\Gamma_3\Gamma_2^2\big(\Gamma_1^2+\Gamma_2^2+\Gamma_3^2\big)(y_1-y_2)(y_3-y_2); && (88) \end{aligned}$$

the third row by

$$\begin{aligned} a_{31} &= a_{13}, && (89)\\ a_{32} &= a_{23}, && (90) \end{aligned}$$

$$a_{33} = -\Gamma_1^2\Gamma_3^2\big(\Gamma_1^2+\Gamma_2^2+\Gamma_3^2\big)(x_1-x_3)^2 - \big(\Gamma_1^2+\Gamma_3^2\big)\big(\Gamma_1^2\Gamma_2^2(y_1-y_2)^2 + \Gamma_1^2\Gamma_3^2(y_1-y_3)^2+\Gamma_2^2\Gamma_3^2(y_2-y_3)^2\big), \tag{91}$$

$$a_{34} = \Gamma_2^2\big(\Gamma_1^2(x_1-x_2)+\Gamma_3^2(-x_2+x_3)\big)\big(\Gamma_1^2(y_1-y_2)+\Gamma_3^2(-y_2+y_3)\big), \tag{92}$$

$$a_{35} = \Gamma_2\Gamma_3\Gamma_1^2\big(\Gamma_1^2+\Gamma_2^2+\Gamma_3^2\big)(x_1-x_2)(x_1-x_3) + \Gamma_2\Gamma_3\big(\Gamma_1^2\Gamma_2^2(y_1-y_2)^2+\Gamma_1^2\Gamma_3^2(y_1-y_3)^2+\Gamma_2^2\Gamma_3^2(y_2-y_3)^2\big), \tag{93}$$

$$a_{36} = \Gamma_2\Gamma_3\big(\Gamma_1^2(x_1-x_2)+\Gamma_3^2(-x_2+x_3)\big)\big(\Gamma_1^2(y_1-y_3)+\Gamma_2^2(y_2-y_3)\big); \tag{94}$$

the fourth row by

$$a_{41} = a_{14}, \tag{95}$$

$$a_{42} = a_{24}, \tag{96}$$

$$a_{43} = a_{34}, \tag{97}$$

$$a_{44} = -\Gamma_1^2\Gamma_3^2\big(\Gamma_1^2+\Gamma_2^2+\Gamma_3^2\big)(y_1-y_3)^2 - \big(\Gamma_1^2+\Gamma_3^2\big)\big(\Gamma_1^2\Gamma_2^2(x_1-x_2)^2 + \Gamma_1^2\Gamma_3^2(x_1-x_3)^2+\Gamma_2^2\Gamma_3^2(x_2-x_3)^2\big), \tag{98}$$

$$a_{45} = \Gamma_2\Gamma_3\big(\Gamma_1^2(x_1-x_3)+\Gamma_2^2(x_2-x_3)\big)\big(\Gamma_1^2(y_1-y_2)+\Gamma_3^2(y_3-y_2)\big), \tag{99}$$

$$a_{46} = \Gamma_2\Gamma_3\big(\Gamma_1^2\Gamma_2^2(x_1-x_2)^2+\Gamma_1^2\Gamma_3^2(x_1-x_3)^2+\Gamma_2^2\Gamma_3^2(x_2-x_3)^2\big) + \Gamma_2\Gamma_3\Gamma_1^2\big(\Gamma_1^2+\Gamma_2^2+\Gamma_3^2\big)(y_2-y_1)(y_3-y_1); \tag{100}$$

the fifth row by

$$a_{51} = a_{15}, \tag{101}$$

$$a_{52} = a_{25}, \tag{102}$$

$$a_{53} = a_{35}, \tag{103}$$

$$a_{54} = a_{45}, \tag{104}$$

$$a_{55} = -\Gamma_1^2\Gamma_2^2\big(\Gamma_1^2+\Gamma_2^2+\Gamma_3^2\big)(x_1-x_2)^2 - \big(\Gamma_1^2+\Gamma_2^2\big)\big(\Gamma_1^2\Gamma_2^2(y_1-y_2)^2 + \Gamma_1^2\Gamma_3^2(y_1-y_3)^2+\Gamma_2^2\Gamma_3^2(y_2-y_3)^2\big), \tag{105}$$

$$a_{56} = \Gamma_3^2\big(\Gamma_1^2(x_1-x_3)+\Gamma_2^2(x_2-x_3)\big)\big(\Gamma_1^2(y_1-y_3)+\Gamma_2^2(y_2-y_3)\big); \tag{106}$$

and the sixth row by

$$a_{61} = a_{16}, \tag{107}$$

$$a_{62} = a_{26}, \tag{108}$$

$$a_{63} = a_{36}, \tag{109}$$

$$a_{64} = a_{46}, \tag{110}$$

$$a_{65} = a_{56}, \tag{111}$$

$$a_{66} = -\Gamma_1^2\Gamma_2^2(\Gamma_1^2+\Gamma_2^2+\Gamma_3^2)(y_1-y_2)^2 - (\Gamma_1^2+\Gamma_2^2)(\Gamma_1^2\Gamma_2^2(x_1-x_2)^2 + \Gamma_1^2\Gamma_3^2(x_1-x_3)^2 + \Gamma_2^2\Gamma_3^2(x_2-x_3)^2). \tag{112}$$

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