B. N. SHASHIKANTH
Mechanical Engineering Department
MSC 3450, PO Box 30001
New Mexico State University
Las Cruces, NM 88003, USA
E-mail: shashi@me.nmsu.edu

POISSON BRACKETS FOR THE DYNAMICALLY INTERACTING SYSTEM OF A 2D RIGID CYLINDER AND N POINT VORTICES: THE CASE OF ARBITRARY SMOOTH CYLINDER SHAPES

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This paper basically extends the work of Shashikanth, Marsden, Burdick and Kelly [17] by showing that the Hamiltonian (Poisson bracket) structure of the dynamically interacting system of a 2-D rigid circular cylinder and N point vortices, when the vortex strengths sum to zero and the circulation around the cylinder is zero, also holds when the cylinder has arbitrary (smooth) shape. This extension is a consequence of a reciprocity relation, obtainable by an application of a classical Green's formula, that holds for this problem. Moreover, even when the vortex strengths do not sum to zero but with the circulation around the cylinder still zero, it is shown that there is a Poisson bracket for the system which differs from the previous bracket by the inclusion of a 2-cocycle term. Finally, comparisons are made to the works of Borisov, Mamaev and Ramodanov [15], [16], [5], [4].

Contents

1. Introduction

The problem of the Hamiltonian structure of the system of a 2-D rigid cylinder interacting dynamically with N point vortices is a problem that, although classical in its formulation, has, apparently, not been well investigated till recently. The problem may be viewed as a blend of two "component" classical

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problems: the Hamiltonian structure of point vortices interacting with a 2-D rigid but stationary boundary, which was solved by C.C.Lin [8], and the Hamiltonian structure of Kirchhoff's equations for the motion of a rigid body (2-D or 3-D) in an ideal fluid without vorticity [12].

Recently, the groups of Shashikanth, Marsden, Burdick and Kelly [17] (SMBK) and Borisov, Mamaev and Ramodanov [15], [16], [5], [4] (BMR), almost simultaneously, addressed this problem by deriving the equations of motion of the finite-dimensional combined cylinder+vortices system and showing the Hamiltonian structure. SMBK showed that at least for the case when the cylinder has a circular shape, the vortex strengths sum to zero and the circulation around the circular cylinder is zero, the equations are Hamiltonian for the total kinetic energy (minus infinite contributions) with respect to a Poisson bracket that is the sum of the Lie-Poisson bracket for Kirchhoff's equations and the canonical point vortex bracket for C.C.Lin's equations. BMR, though essentially following the same momentum balance approach, came up with a different set of Poisson brackets owing to the choice of a different set of coordinates on their phase space.

In this paper, it is shown that the SMBK bracket structure holds, under the conditions of zero total vortex strength and zero cylinder circulation, even if the cylinder shape is arbitrary (though smooth) and that this result relies on a reciprocity relation obtainable by an application of one of the integral formulas of Green in calculus. Moreover, it is shown that even if the vortex strengths do not sum to zero but with the circulation around the cylinder still zero then the dynamical system is still Hamiltonian. The Poisson brackets in this case are obtained by extending the SMBK bracket by the addition of a 2-cocycle term. The organization of the paper is as follows: In section 2, the SMBK equations are reviewed. In section 3, the Poisson brackets for the case of arbitrary (smooth) cylinder shapes are presented for the case when the sum of the vortex strengths $\Gamma = 0$ and the circulation around the cylinder is zero. In section 4, the brackets for the case $\Gamma \neq 0$ but with the circulation around the cylinder still zero are presented. In the final section 5, in an attempt to link the works of the SMBK and BMR groups, comparisons are made between the equations and bracket structures obtained by the two groups and it is shown that they are essentially the same.

Finally, it should be reiterated that the emphasis in this paper, as in [17], is on the equations of motion of the *coupled* rigid boundary+vortices system and to investigate their Hamiltonian structure. The goal is not to obtain expressions for forces and moments due to the fluid on the rigid boundary, as has been attempted in a few other papers, but to set the basis for further investigations of the dynamics of this coupled system and with control inputs added. Dynamics of rigid bodies interacting with smooth vorticity fields on extended domains have been investigated, as coupled infinite-dimensional systems, to some extent in [18], [19], [7] and [14] but there still remains much to be done in the infinite-dimensional setting; see [17] for other references.

2. Background: SMBK equations and bracket

In [17], the authors showed that the equations of motion of a 2-D rigid cylinder of arbitrary (smooth) shape dynamically interacting with N point vortices, as depicted in Fig. 1, when the vortex strengths sum to zero and the circulation around the cylinder is zero, can be written as:[1]

$$\left(\frac{d}{dt} + \mathbf{\Omega}\times\right)\mathbf{L} = 0, \tag{2.1}$$

$$\frac{d\mathbf{A}}{dt} + \mathbf{V} \times \mathbf{L} = 0, \tag{2.2}$$

$$\Gamma_j \left(\frac{d\mathbf{l}_j}{dt} + \mathbf{\Omega} \times \mathbf{l}_j + \mathbf{V}\right) = J\left(\frac{\partial W}{\partial \mathbf{l}_j}\right), j = 1, \ldots, N, \tag{2.3}$$

[1] All quantities in the equations are with reference to a body-fixed frame whose origin is at the body center-of-mass and whose axes coincide with the principal axes of inertia.

where $\mathbf{V}$ is the velocity of the body center of mass, $\mathbf{\Omega}$ is the body rotational velocity, $\mathbf{L}$ and $\mathbf{A}$ are the linear and angular momenta of the system (i. e. the impulse of the fluid plus the momentum of the body), respectively, given by:

$$\begin{pmatrix} \mathbf{L} \\ \mathbf{A} \end{pmatrix} = M \begin{pmatrix} \mathbf{V} \\ \mathbf{\Omega} \end{pmatrix} + \begin{pmatrix} \mathbf{p} \\ \pi \end{pmatrix}, \tag{2.4}$$

where

$$\mathbf{p} = \sum \Gamma_j \mathbf{l}_j \times \mathbf{k} + \oint_{\partial B} \mathbf{l} \times (\mathbf{n} \times \mathbf{u}_V)\, ds \tag{2.5}$$

and

$$\pi = -\frac{1}{2} \sum \Gamma_j \langle \mathbf{l}_j, \mathbf{l}_j \rangle \mathbf{k} - \frac{1}{2} \oint_{\partial B} l^2 (\mathbf{n} \times \mathbf{u}_V)\, ds \tag{2.6}$$

Here, M is a 3×3 *symmetric mass tensor* that depends only on the body shape and body mass, $\mathbf{l}$ is the position vector of a point (with $l^2 = \langle \mathbf{l}, \mathbf{l} \rangle$), $\mathbf{l}_j$ is the position vector of the jth point vortex in the body-fixed frame and J is the matrix:

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The contour integrals are around the body boundary ∂B and $\mathbf{n}$ denotes the unit (into the fluid) normal. The term $\mathbf{u}_V$ denotes the divergence-free, body-parallel component of the total velocity field and is the sum of the velocity field due to the N external vortices and the (potential) velocity field due to the image vorticity. Observe that the contour integrals depend only on the shape of the body, the strength and the position of the vortices (in the body-fixed frame).

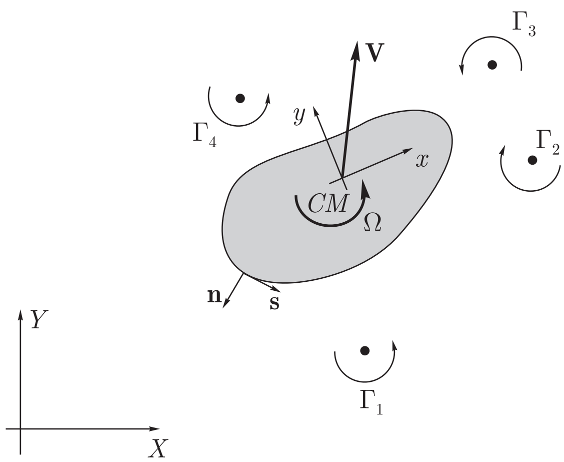

Fig. 1. 2-D rigid cylinder of arbitrary (but smooth) shape dynamically interacting with N point vortices external to it

W is the Kirchhoff–Routh function [8] generalized to moving boundaries [17] and given by:

$$\begin{aligned} W(\mathbf{l}_j, \mathbf{V}(t), \mathbf{\Omega}(t)) &= \sum \Gamma_j \psi_B(\mathbf{l}_j, \mathbf{V}(t), \mathbf{\Omega}(t)) + W_G(\mathbf{l}_j) = \\ &= \sum \Gamma_j \psi_B(\mathbf{l}_j, \mathbf{V}(t), \mathbf{\Omega}(t)) + \sum_{k,j(k>j)} \Gamma_k \Gamma_j G(\mathbf{l}_k; \mathbf{l}_j) + \\ &+ \frac{1}{2} \sum \Gamma_j^2 g(\mathbf{l}_j; \mathbf{l}_j) \end{aligned} \tag{2.7}$$

In the above, ψ_B is the stream function of the Kirchhoff flow associated with the motion of the body, G is a Green's function satisfying appropriate boundary conditions and of the form

$$G(x, y; x_0, y_0) = g(x, y; x_0, y_0) + \frac{1}{4\pi} \log[(x - x_0)^2 + (y - y_0)^2], \tag{2.8}$$

and g is harmonic everywhere in the fluid domain and is the stream function of the irrotational velocity field of the image vorticity which annuls the non-zero normal velocities on the body due to the external vortices. All three functions G, g and ψ_B depend on the body shape and, moreover, G and g are symmetric. For a circle g, and hence G, can be computed using Milne-Thomson's circle theorem and ψ_B is known from classical potential theory for fluids [12]. For some non-trivial shapes these functions can be computed using conformal mapping techniques. It should be noted that for swimming objects like fish, an airfoil shape is perhaps the most appropriate, and there exists a conformal map from the exterior of a circle to the exterior of a Juokowski airfoil.

The stream function ψ_B, for arbitrary body shape, can be written as:

$$\psi_B(\mathbf{l}_j, \mathbf{V}(t), \mathbf{\Omega}(t)) = \mathbf{V}(t) \cdot \eta(\mathbf{l}_j) + \mathbf{\Omega}(t) \cdot \kappa(\mathbf{l}_j) = \langle (\eta(\mathbf{l}_j), \kappa(\mathbf{l}_j)), (\mathbf{V}(t), \mathbf{\Omega}(t)) \rangle . \tag{2.9}$$

The fields $\eta(\mathbf{l})$ (of 2-vectors) and $\kappa(\mathbf{l})$ (of 1-vectors) depend only on the shape of the body. Their components are the harmonic conjugates of the Kirchhoff potentials that appear in the analogous linear decomposition of the potential function of the Kirchhoff flow:

$$\Phi_B(\mathbf{l}, \mathbf{V}(t), \mathbf{\Omega}(t)) = \mathbf{V}(t) \cdot \phi(\mathbf{l}) + \mathbf{\Omega}(t) \cdot \xi(\mathbf{l}) = \tag{2.10}$$

$$= V_x \phi_x + V_y \phi_y + \Omega \xi, \tag{2.11}$$

where the functions ϕ_x, ϕ_y and ξ are unit potential functions harmonic in the fluid domain, have vanishing gradients at infinity and satisfy the following body boundary conditions:

$$\frac{\partial \phi_x}{\partial n} = n_x \qquad \frac{\partial \phi_y}{\partial n} = n_y \qquad \frac{\partial \xi}{\partial n} = n_y x - n_x y, \tag{2.12}$$

where $\mathbf{n} = n_x \mathbf{i} + n_y \mathbf{j}$. Denoting the unit tangent vector on the body boundary by $\mathbf{s} = s_x \mathbf{i} + s_y \mathbf{j}$, the corresponding boundary conditions satisfied by the components of ψ_B are obtained by using the relations $\partial \phi_x / \partial n = -\partial \eta_x / \partial s$ etc., and $n_x = s_y$, $n_y = -s_x$.

$$\frac{\partial \eta_x}{\partial s} = -s_y \qquad \frac{\partial \eta_y}{\partial s} = s_x \qquad \frac{\partial \kappa}{\partial s} = s_x x + s_y y. \tag{2.13}$$

3. Hamiltonian structure for the case of arbitrary shapes

In this section, a classical formula due to Green is applied to a dynamically interacting rigid body-vorticity field system. It is then shown that this leads to a reciprocity relation that holds for the problem being considered. Finally, the reciprocity relation is used to prove the Hamiltonian structure for the case of arbitrary cylinder shapes.

3.1. Green's formula for rigid body-vorticity field systems in the plane

Consider smooth vorticity fields $\omega(\mathbf{l}, t)$ in the fluid domain D with boundary ∂B which coincides with the boundary of the moving body. The stream function of the fluid flow is:

$$\psi(\mathbf{l}, t) = \psi_V(\mathbf{l}, t) + \psi_B(\mathbf{l}, t),$$

where ψ_V is the streamfunction of the velocity field $\mathbf{u}_V$ and satisfies

$$\nabla^2 \psi_V(\mathbf{l}, t) = \omega(\mathbf{l}, t), \qquad \psi_V = \text{constant} \quad \text{on} \quad \partial B,$$

and

$$\nabla^2 \psi_B(\mathbf{l}, t) = 0, \qquad \psi_B = \mathbf{V}(t) \cdot \eta(\mathbf{l}) + \mathbf{\Omega}(t) \cdot \kappa(\mathbf{l}) \quad \text{on} \quad \partial B \cup D. \tag{3.1}$$

There is also the following (zero-circulation) condition satisfied by each of ψ_B and ψ_V:

$$\oint_{\partial B} \frac{\partial \psi_B}{\partial n} ds = 0, \qquad \oint_{\partial B} \frac{\partial \psi_V}{\partial n} ds = 0, \tag{3.2}$$

A straightforward application of one of the classical integral formulas of Green leads to:

$$\begin{aligned}\int_D \psi_B \omega dA &= \int_D \psi_B \nabla^2 \psi_V dA = \\ &= \int_D \psi_V \nabla^2 \psi_B dA + \oint_{\partial B + C_R} \psi_V \frac{\partial \psi_B}{\partial n} ds - \oint_{\partial B + C_R} \psi_B \frac{\partial \psi_V}{\partial n} ds = \\ &= -\oint_{\partial B} \psi_B \frac{\partial \psi_V}{\partial n} ds, \end{aligned} \tag{3.3}$$

using (3.1) and the boundary conditions. The boundary terms on an imaginary closed external boundary C_R of mean radius R can be shown to behave as follows as $R \to \infty$:

$$\oint_{C_R} \psi_V \frac{\partial \psi_B}{\partial n} ds = O(\log(R)/R),$$

$$\oint_{C_R} \psi_B \frac{\partial \psi_V}{\partial n} ds = O(1/R),$$

and hence vanish in the limit.

The condition (3.3) may also be viewed as an L^2-orthogonality condition between the space of vorticities and the space of stream functions of potential flows. Indeed it is obtainable from the L^2-orthogonality relation for the Hodge-Helmholtz components of the velocity vector field using integration by parts. The space of vorticities, in this non-simply-connected domain, is the set of vorticities supplemented by tangential velocity one-form densities around non-contractible loops [11]. Discussions of projection conditions like this and others in a more abstract framework of potential theory and harmonic forms may be found, for example, in the papers of Weyl [22] and Duff and Spencer [6].

3.2. Reciprocity relation

The following will now be shown to be true. It should be mentioned that similar results have also been worked out by Wells [20],[21] for vortex filaments interacting with a sphere but without reference to the Hamiltonian dynamics of the interacting system.

Lemma 3.1. *For the dynamically interacting system of a 2-D rigid cylinder, with zero circulation around it, and N point vortices external to it, the following relations hold irrespective of the shape of cylinder and the sum of the strengths of the vortices:*

$$\left(\sum \Gamma_j \eta(\mathbf{l}_j), \sum \Gamma_j \kappa(\mathbf{l}_j)\right) = \left(\oint_{\partial B} \mathbf{l} \times (\mathbf{n} \times \mathbf{u}_V)\, ds, -\frac{1}{2} \oint_{\partial B} l^2\, (\mathbf{n} \times \mathbf{u}_V)\, ds\right) \tag{3.4}$$

Proof. First, we note that:

$$\mathbf{n} \times \mathbf{u}_V = (\mathbf{u}_V \cdot \mathbf{s})\, \mathbf{k} = \frac{\partial \psi_V}{\partial n} \mathbf{k}.$$

From (3.3), and noting the linear decomposition of ψ_B (2.9), we see that the proof just requires that the following hold:

$$\eta(\mathbf{l})_{|\partial B} = J\mathbf{l} + \text{constant},$$

$$\kappa(\mathbf{l})_{|\partial B} = \frac{1}{2} l^2 + \text{constant}.$$

The constant terms do not affect the result since, by (3.2), $\oint_{\partial B} \mathbf{n} \times \mathbf{u}_V\, ds = 0$. Consider now the following three functions defined on $\mathbb{R}^2$:

$$\tilde{\eta}_x(\mathbf{l}) = -y,$$

$$\tilde{\eta}_y(\mathbf{l}) = x,$$

$$\tilde{\kappa}(\mathbf{l}) = \frac{x^2 + y^2}{2}.$$

Their gradients in the tangential direction on ∂B are:

$$\frac{\partial \tilde{\eta}_x}{\partial s} = -s_y \qquad \frac{\partial \tilde{\eta}_y}{\partial s} = s_x \qquad \frac{\partial \tilde{\kappa}}{\partial s} = s_x x + s_y y, \tag{3.5}$$

or, in other words, they satisfy the same boundary conditions as (2.13). If two maps have the same derivative map at each point of a domain then they can differ at most by a constant when restricted to that domain. Explicitly, let $i : \partial B \to \mathbb{R}^2$ represent the inclusion map and i^{-1} its inverse (defined on the range of i). The equality of the tangential derivatives means $\mathbf{D}i^{-1} \circ \mathbf{D}\tilde{\eta}_x = \mathbf{D}i^{-1} \circ \mathbf{D}\eta_x$ and so on, where $\mathbf{D}$ denotes the derivative map. By the chain rule this implies that $\mathbf{D}\left(i^{-1} \circ \tilde{\eta}_x \mid_{\partial B}\right) = \mathbf{D}\left(i^{-1} \circ \eta_x \mid_{\partial B}\right)$ and so on. It follows that

$$\eta(\mathbf{l})_{|\partial B} = \tilde{\eta}(\mathbf{l})_{|\partial B} + \text{constant},$$
$$\kappa(\mathbf{l})_{|\partial B} = \tilde{\kappa}(\mathbf{l})_{|\partial B} + \text{constant}.$$

It should be pointed out that the above feature of the stream function associated with the Kirchhoff flow is also mentioned in [12], §9.40.

From the above, one obtains that

$$-\oint_{\partial B} (\eta, \kappa) \frac{\partial \psi_V}{\partial n} ds = \left(\oint_{\partial B} \mathbf{l} \times (\mathbf{n} \times \mathbf{u}_V)\ ds, -\frac{1}{2} \oint_{\partial B} l^2 (\mathbf{n} \times \mathbf{u}_V)\ ds \right)$$

The statement of the lemma then follows from (3.3) which can be extended to a point vortex distribution by applying it to the fluid domain with circles of radius ϵ centered on each point vortex and letting $\epsilon \to 0$. This is the same as substituting $\omega = \sum \Gamma_j \delta(\mathbf{l}_j - \mathbf{l})$ directly in the left hand side of (3.3). ■

The reciprocity relation (3.4) is reminiscent of relations in elasticity, for example Betti's theorem; the reader is referred to [9] for a more detailed discussion of reciprocity relations in that field.

3.3. Poisson brackets for the system

Using the reciprocity relation of Lemma 3.1, the Hamiltonian structure for arbitrary shapes will now be shown.

Consider the following choice of the Hamiltonian of the system (2.1), (2.2) and (2.3) in the canonical variables $\mathbf{L}$, $\mathbf{A}$ and $\mathbf{l}_j$ $(j = 1, ..., N)$:

$$\begin{aligned} H &= -W_G(\mathbf{l}_j) + \frac{1}{2} (\mathbf{V}, \boldsymbol{\Omega})^T M (\mathbf{V}, \boldsymbol{\Omega}) = \\ &= -W_G(\mathbf{l}_j) + \frac{1}{2} (\mathbf{L} - \mathbf{p}, \mathbf{A} - \pi)^T (M^{-1})^T M M^{-1} (\mathbf{L} - \mathbf{p}, \mathbf{A} - \pi) = \\ &= -W_G(\mathbf{l}_j) + \frac{1}{2} (\mathbf{L} - \mathbf{p}, \mathbf{A} - \pi)^T M^{-1} (\mathbf{L} - \mathbf{p}, \mathbf{A} - \pi) = \\ &= -W_G(\mathbf{l}_j) + \frac{1}{2} [(\mathbf{L}, \mathbf{A}) - (\mathbf{p}, \pi)]^T M^{-1} [(\mathbf{L}, \mathbf{A}) - (\mathbf{p}, \pi)] = \\ &= -W_G(\mathbf{l}_j) + \frac{1}{2} \left[(\mathbf{L}, \mathbf{A})^T M^{-1} (\mathbf{L}, \mathbf{A}) + (\mathbf{p}, \pi)^T M^{-1} (\mathbf{p}, \pi) - 2 (\mathbf{p}, \pi)^T M^{-1} (\mathbf{L}, \mathbf{A}) \right] \end{aligned} \tag{3.6}$$

This Hamiltonian H is the kinetic energy of the body+fluid system *minus* infinite contributions. These contributions arise due to two standard reasons (see, for example, [3], §7.3): (i) the singular nature of the velocity field of the point vortices and (ii) the fact that the flow domain is unbounded. The contribution arising due to (ii) is absent if $\Gamma = 0$.

Lemma 3.2. *The system of equations* (2.1), (2.2) *and* (2.3) *with Hamiltonian function* (3.6) *is a Poisson vector field on the space* $P = \mathfrak{se}(2)^* \times (\mathbb{R}^{2N} \backslash (\triangle \cup B^N)) \equiv P_b \times P_v$ *equipped with the following Poisson bracket. For* $F, G : P \to \mathbb{R}$, *define*

$$\{F, G\}_P = \{F\mid_{P_b}, G\mid_{P_b}\}_{\text{Lie-Poisson}} + \{F\mid_{P_v}, G\mid_{P_v}\}_{\text{point vortex}}.$$

Therefore, if $p(t) = (\mu(t), \mathbf{l}_j(t)) \in P$ *is an integral curve of the system, where* $\mu(t) = (\mathbf{L}(t), \mathbf{A}(t))$, *then*

$$\frac{dF}{dt} := \left\langle \nabla_p F, \frac{dp}{dt} \right\rangle = \left\langle \nabla_\mu F, \mathrm{ad}^*_{\partial H/\partial \mu}\, \mu \right\rangle + \sum_{j=1}^{N} \left\langle \nabla_j F, J^{-1} \nabla_j (H/\Gamma_j) \right\rangle.$$

Proof. First, the equations (2.1) and (2.2) will be shown to be of Lie-Poisson form. The Lie–Poisson equations[6] on $\mathfrak{se}(2)^*$, the dual of the Lie algebra of the Euclidean group on the plane SE(2), are given by:

$$\frac{d\mu}{dt} = \mathrm{ad}^*_{\delta h/\delta \mu}\, \mu,$$

for $h : \mathfrak{se}(2)^* \to \mathbb{R}$. In this problem the operator ad^* takes the form:

$$\mathrm{ad}^*_{(\partial H/\partial \mathbf{A}, \partial H/\partial \mathbf{L})}(\mathbf{A}, \mathbf{L}) = \left(- \left\langle \mathbf{L}, J \frac{\partial H}{\partial \mathbf{L}} \right\rangle, -\frac{\partial H}{\partial A} J \mathbf{L} \right).$$

It is easy to check from (3.6) that

$$\frac{\partial H}{\partial \mathbf{A}} = \mathbf{\Omega}, \tag{3.7}$$

$$\frac{\partial H}{\partial \mathbf{L}} = \mathbf{V}. \tag{3.8}$$

so that

$$\mathrm{ad}^*_{(\partial H/\partial \mathbf{A}, \partial H/\partial \mathbf{L})}(\mathbf{A}, \mathbf{L}) = (-\mathbf{V} \times \mathbf{L}, -\mathbf{\Omega} \times \mathbf{L}),$$

which proves the Lie-Poisson part of the theorem. Now to show that (2.3) are canonical point vortex, compute:

$$\begin{aligned}
\frac{\partial H}{\partial \mathbf{l}_j} &= -\frac{\partial W_G}{\partial \mathbf{l}_j} + \left(\frac{\partial \mathbf{p}}{\partial \mathbf{l}_j}, \frac{\partial \pi}{\partial \mathbf{l}_j} \right) \left[M^{-1}(\mathbf{p}, \pi) - M^{-1}(\mathbf{L}, \mathbf{A}) \right] = \\
&= -\frac{\partial W_G}{\partial \mathbf{l}_j} - \left(\frac{\partial \mathbf{p}}{\partial \mathbf{l}_j}, \frac{\partial \pi}{\partial \mathbf{l}_j} \right) (\mathbf{V}, \mathbf{\Omega}) = \\
&= -\frac{\partial W_G}{\partial \mathbf{l}_j} - \begin{pmatrix} 0 & -\Gamma_j & -\Gamma_j x_j \\ \Gamma_j & 0 & -\Gamma_j y_j \end{pmatrix} (\mathbf{V}, \mathbf{\Omega}) - \\
&\quad - \left\langle \frac{\partial}{\partial \mathbf{l}_j} \left(\oint_{\partial B} \mathbf{l} \times (\mathbf{n} \times \mathbf{u}_V)\, ds, -\frac{1}{2} \oint_{\partial B} l^2 (\mathbf{n} \times \mathbf{u}_V)\, ds \right), (\mathbf{V}, \mathbf{\Omega}) \right\rangle
\end{aligned}$$

Using Lemma 3.1 and (2.7) it follows that:

$$J \frac{\partial H}{\partial \mathbf{l}_j} = -J \frac{\partial W}{\partial \mathbf{l}_j} + \Gamma_j \mathbf{V} + \Gamma_j \mathbf{\Omega} \times \mathbf{l}_j,$$

and that (2.3) is of the form

$$\Gamma_j \frac{d\mathbf{l}_j}{dt} = -J \frac{\partial H}{\partial \mathbf{l}_j}, \quad j = 1, \ldots, N$$

■

Note: At this stage, a small error in [17] that was unfortunately not corrected in the erratum to that paper either should be pointed out. The point vortex bracket in [17] was inadvertently written with a sign opposite to that in the above Lemma.

4. Momentum maps and the case of $\sum \boldsymbol{\Gamma}_j \neq \mathbf{0}$

The case when the vortex strengths do not sum to zero but with the circulation around the cylinder still zero will now be discussed. In this case, equations (2.1), (2.2) and (2.3) get slightly modified by the presence of additional terms in the equation for the linear momentum $\mathbf{L}$. The equations read:

$$\left(\frac{d}{dt} + \boldsymbol{\Omega}\times\right)\mathbf{L} = \Gamma\mathbf{k}\times\mathbf{V}, \tag{4.1}$$

$$\frac{d\mathbf{A}}{dt} + \mathbf{V}\times\mathbf{L} = 0, \tag{4.2}$$

$$\Gamma_j\left(\frac{d\mathbf{l}_j}{dt} + \boldsymbol{\Omega}\times\mathbf{l}_j + \mathbf{V}\right) = J\left(\frac{\partial W}{\partial \mathbf{l}_j}\right), j = 1, \ldots, N, \tag{4.3}$$

where $\Gamma = \sum \Gamma_j$. Note that in [17], these equations were written in a slightly different form with the position of the cylinder center of mass $\mathbf{a}$ introduced as an additional coordinate, and with $\mathbf{L}, \mathbf{A}$ redefined to include terms involving $\mathbf{a}$.

The Hamiltonian structure of (4.1), (4.2) and (4.3) follows from a result related to the central extension of Lie algebras which is stated in Abraham and Marsden [1], and which has also been applied by Adams and Ratiu [2] for the *unbounded* three point vortex problem. Adams and Ratiu show that in the unbounded problem, the momentum map [10] of the diagonal SE(2) action on the phase space of the N vortices (in the unbounded plane) is not Ad*-equivariant (which, roughly speaking, means that the momentum map does not commute with the group action [1], [10]) when $\Gamma \neq 0$. The theory says that in such a case the symmetry reduced Poisson manifold inherits a Poisson bracket that has an additional term induced by the central extension of $\mathfrak{se}(2)$ by $\mathfrak{co}$, where $\mathfrak{co} : \mathfrak{se}(2) \times \mathfrak{se}(2) \to \mathbb{R}$ is a real-valued 2-cocycle of the Lie algebra $\mathfrak{se}(2)$.

In this paper, where there is a moving rigid boundary, momentum maps and group actions have not been investigated. However, it may be reasonably conjectured that the SE(2)-invariant system, (4.1), (4.2) and (4.3), can be obtained from symmetry reduction of a (unreduced) system with SE(2) symmetry and has a momentum map that is not Ad*-equivariant when $\Gamma \neq 0$. Physically speaking, it can be argued that the momentum map must be the spatial momentum of the body+fluid system and must contain fluid impulse terms like $\sum \Gamma_j \mathbf{r}_j \times \mathbf{k}$ and $-\frac{1}{2}\sum \Gamma_j \langle \mathbf{r}_j, \mathbf{r}_j \rangle \mathbf{k}$, where $\mathbf{r}_j$ is the position vector of the jth vortex in a spatially- fixed frame, and since these are the same terms that lead to the lack of Ad*-equivariance, when $\Gamma \neq 0$, in the unbounded problem they are likely to play the same role here.

Indeed, if we consider the 2-cocycle of the unbounded N vortex problem [2],

$$\mathfrak{co}\left((\theta, \mathbf{a}), (\phi, \mathbf{b})\right) = \Gamma(a_y b_x - a_x b_y)$$

where

$$(\theta, \mathbf{a}), (\phi, \mathbf{b}) \in \mathbb{R}\times\mathbb{R}^2 \equiv \mathfrak{se}(2)\times\mathfrak{se}(2),$$

which can be viewed as a Poisson bracket on $\mathfrak{se}^*(2)$ by making the identification

$$\partial\hat{F}/\partial\mathbf{L} \equiv \mathbf{a}, \quad \partial\hat{G}/\partial\mathbf{L} \equiv \mathbf{b},$$

for $\hat{F}, \hat{G} : \mathfrak{se}^*(2)) \to \mathbb{R}$ and where $(A, \mathbf{L}) \in \mathfrak{se}^*(2) \equiv \mathbb{R}\times\mathbb{R}^2$, then we obtain

Lemma 4.1. *The system of equations* (4.1), (4.2) *and* (4.3) *with Hamiltonian function* (3.6) *is a Poisson vector field on the space* $P = \mathfrak{se}(2)^* \times (\mathbb{R}^{2N} \backslash (\triangle \cup B^N)) \equiv P_b \times P_v$ *equipped with the following Poisson bracket. For* $F, G : P \to \mathbb{R}$, *define*

$$\{F, G\}_P = \{F\mid_{P_b}, G\mid_{P_b}\}_{\text{Lie-Poisson}} + \{F\mid_{P_b}, G\mid_{P_b}\}_{2-\text{cocycle}} + \{F\mid_{P_v}, G\mid_{P_v}\}_{\text{point vortex}}, \tag{4.4}$$

where, for $\hat{F}, \hat{G} : \mathfrak{se}^*(2) \to \mathbb{R}$,

$$\{\hat{F}, \hat{G}\}_{2-\text{cocycle}} := \Gamma \left(\frac{\partial \hat{F}}{\partial L_y} \frac{\partial \hat{G}}{\partial L_x} - \frac{\partial \hat{F}}{\partial L_x} \frac{\partial \hat{G}}{\partial L_y} \right)$$

Proof. In addition to what has been already proved in Lemma 3.2, the proof just requires showing that the extra terms in (4.1) arise from the 2-cocycle bracket. This is a straightforward check using (3.8). ■

5. The relation between the SMBK and BMR$_{\Gamma^*=\Gamma}$ equations and brackets

In this final section, an attempt is made to link the works of SMBK [17] and BMR [15], [16], [5], [4]. The equations and Poisson brackets derived by SMBK and BMR for the case of a circular cylinder are compared. It is shown that the two groups essentially obtained the same results though approaching the problem in somewhat different ways.

However, due to some notational inconsistency in the papers by BMR some assumptions need to be made before making the comparison and these are discussed below:

1. In Ramodanov [16], the author uses the parameter Γ^*, which is characterized by the phrase “the constant $\Gamma^* - \Gamma$ is a measure of fluid circulation around” the cylinder (p.292 of cited reference). From this, one can conclude that

$$\Gamma^* = \Gamma + \text{circulation around cylinder} \tag{5.1}$$

   This constant Γ^* appears in Ramodanov’s final equation for the motion of the cylinder (equation (11) in the cited reference) as the term $i\Gamma^* \rho dk_0/dt$ which, written in our notation, is the term $\rho \Gamma^* \mathbf{k} \times \mathbf{V}$. Unfortunately, in the later papers [5], [4], the authors introduce new notation λ instead of Γ^* and, moreover, this λ, apparently, denotes (modulo a factor of $1/2\pi$) *only* the circulation around the cylinder. Thus, in the cylinder equations in these later references, the term corresponding to $\rho \Gamma^* \mathbf{k} \times \mathbf{V}$ which is $\mathbf{V} \times \lambda \mathbf{k}$ does not seem to account for the total strength of the N vortices i.e (in our notation) the term Γ.

   Since we believe Ramodanov’s derivation [16] is correct, we make our first assumption as follows:

   Assumption A: In [5], [4], the definition of the term λ is assumed to be:

$$\lambda = \frac{\Gamma^*}{2\pi},$$

   where Γ^* is given by (5.1).

2. There is also some inconsistency in notation for the mass and added mass of the cylinder. In the term $\mu \dot{v}_1$ appearing in equation (1.1) of [5], μ is defined as the mass of the cylinder. But exactly the same term is written as $a \dot{v}_1$ in equation (1.1) of [4] and there a is defined as a coefficient

which "involves the added mass of the cylinder". The added mass is never written explicitly anywhere. Note that in SMBK, for the case of the circular cylinder, the term $c = m + \pi R^2$ denotes the mass plus added mass of the cylinder (per unit length perpendicular to the plane), assuming a value of fluid and body density of 1.

Further, in [4], it is stated that "the density of the fluid is 2π". It should be noted that in a neutrally buoyant situation, the density of the fluid and the body are the same and therefore cancel out in the cylinder equations. The density appears only in the Hamiltonian kinetic energy. It is difficult to explain the factor of $1/2\pi$ that appears on the right of the cylinder equations in [5], [4]. Therefore, our second assumption is

Assumption B: The term μ in [5] and the term c in [17] are related as follows:

$$\mu = \frac{c}{2\pi}$$

3. Finally, it should be noted that

Fact A: BMR follow a different sign convention from SMBK for vortex strenghts/circulations. BMR assume that *clockwise* circulations are positive, whereas SMBK assume *counterclockwise* circulations are positive.

Definition: Equations (1.1) in [5] with Assumptions A and B in force and with zero circulation around the cylinder, will be defined as the $\mathrm{BMR}_{\Gamma^*=\Gamma}$ equations.

5.1. SMBK and $\mathbf{BMR_{\Gamma^*=\Gamma}}$ equations

Recall that the SMBK equations for the case of a circular cylinder and point vortices are:

$$\frac{d\mathbf{L}}{dt} = \Gamma\mathbf{k} \times \mathbf{V} \tag{5.2}$$

$$\Gamma_j \frac{d\mathbf{l}_j}{dt} = -\Gamma_j \mathbf{V} + J\frac{\partial W}{\partial \mathbf{l}_j}, \quad j = 1, \ldots, N, \tag{5.3}$$

which are obtained from (4.1) and (4.3), respectively, by putting $\mathbf{\Omega} = 0$, and ignoring equation (4.2) which gets decoupled from the above. The linear momentum $\mathbf{L}$ of the system (i.e. fluid linear impulse plus cylinder linear momentum) is given by:

$$\mathbf{L} = c\mathbf{V} + \sum \Gamma_j \mathbf{l}_j \times \mathbf{k} + R^2 \sum \mathbf{k} \times \Gamma_j \left(\frac{x_j}{x_j^2 + y_j^2}, \frac{y_j}{x_j^2 + y_j^2} \right), \tag{5.4}$$

The cylinder equations: Rewriting (5.2) as an equation for the cylinder center of mass velocity $\mathbf{V}$ (using (5.4)), we obtain

$$c\frac{d\mathbf{V}}{dt} = \Gamma\mathbf{k} \times \mathbf{V} + \sum \Gamma_j \left(\frac{d\tilde{\mathbf{l}}_j}{dt} - \frac{d\mathbf{l}_j}{dt} \right) \times \mathbf{k}, \tag{5.5}$$

where $\tilde{\mathbf{l}}_j = \left(R^2 / \mid \mathbf{l}_j \mid^2 \right) \mathbf{l}_j$ are the coordinates of the image vortex at the inverse point of the jth vortex.

Now rewrite the $\mathrm{BMR}_{\Gamma^*=\Gamma}$ equations in the notation of this paper. The $\mathrm{BMR}_{\Gamma^*=\Gamma}$ equations for $\mathbf{V}$ are:

$$c\frac{d\mathbf{V}}{dt} = \mathbf{V} \times \Gamma\mathbf{k} + \sum \Gamma_j \left(\frac{d\mathbf{l}_j}{dt} - \frac{d\tilde{\mathbf{l}}_j}{dt} \right) \times \mathbf{k}, \tag{5.6}$$

Comparison of (5.5) with (5.6), keeping in mind Fact A, shows that they are the same.

The point vortex equations: To compare the point vortex equations, let us first write down the expanded form of (5.3). The Kirchhoff-Routh function W for the case of a circular cylinder is given by (2.7) with

$$\psi_B\left(\mathbf{l}_k, \mathbf{V}\right) = R^2 \left\langle \left(\frac{-y_k}{x_k^2+y_k^2}, \frac{x_k}{x_k^2+y_k^2} \right), \mathbf{V} \right\rangle = \tag{5.7}$$

$$g\left(\mathbf{l}_k, \mathbf{l}_k\right) = \frac{1}{4\pi}\left(\log(x_k^2+y_k^2) - \log\left(\left(x_k - \frac{R^2 x_k}{l_k^2} \right)^2 + \left(y_k - \frac{R^2 y_k}{l_k^2}\right)^2 \right)\right),$$

$$= -\frac{1}{2\pi}\log\left(1 - \frac{R^2}{l_k^2}\right), \tag{5.8}$$

$$G\left(\mathbf{l}_k, \mathbf{l}_j\right) = g\left(\mathbf{l}_k, \mathbf{l}_j\right) + \frac{1}{4\pi}\log((x_k-x_j)^2+(y_k-y_j)^2) =$$

$$= \frac{1}{4\pi}\left(\log(x_k^2+y_k^2) - \log\left(\left(x_k - \frac{R^2 x_j}{l_j^2} \right)^2 + \left(y_k - \frac{R^2 y_j}{l_j^2}\right)^2 \right) + \log((x_k-x_j)^2+(y_k-y_j)^2)\right), \tag{5.9}$$

where $l_j^2 = |\, \mathbf{l}_j \,|^2 = x_j^2 + y_j^2$. The terms in G are the streamfunctions due to the jth vortex ($j \neq k$) and its two images evaluated at the location of the kth vortex . The terms in g are the streamfunctions due to the two images of the kth vortex. And ψ_B is the streamfunction at the location of the kth vortex due to the Kirchhoff flow associated with the translation of the cylinder. Thus, W gives the streamfunction of the total flow minus the kth point vortex and therefore, as per Routh's rule, $(1/\Gamma_k)J\partial W/\partial \mathbf{l}_k$ will give the velocity of the kth vortex in a fixed reference frame. Subtracting the cylinder velocity gives the velocity of the kth vortex in the body-fixed frame and equation (2.3). Note that C.C.Lin, in his original work, introduced the factor of $1/2$ in front of $g(\mathbf{l}_k, \mathbf{l}_k)$ as it appears in W to account for the fact that the coordinate of the inverse image vortex is written directly in terms of the coordinates of the external vortex and hence differentiating g with respect to $\mathbf{l}_k$ gives twice the resultant velocity due to the two images.

BMR also use Routh's rule but use velocity potentials instead of stream functions. Their velocity potential (equation (1.2) in [5]), evaluated at the location of the kth vortex, due to the total flow minus the kth vortex is:

$$\phi(\mathbf{l}_k) = -R^2 \left\langle \left(\frac{x_k}{x_k^2+y_k^2}, \frac{y_k}{x_k^2+y_k^2} \right), \mathbf{V} \right\rangle + \frac{\Gamma}{2\pi}\tan^{-1}\left(\frac{y_k}{x_k}\right) + \sum_{j=1}^{N} \frac{\Gamma_j}{2\pi}\tan^{-1}\left(\frac{y_k - \tilde{y}_j}{x_k - \tilde{x}_j}\right) -$$
$$- \sum_{j=1, j\neq k}^{N} \frac{\Gamma_j}{2\pi}\tan^{-1}\left(\frac{y_k - y_j}{x_k - x_j}\right)$$

Noting Fact A again, it is easy to verify that the BMR equations for the kth point vortex (equation (1.1) in [5]),

$$\frac{d\mathbf{l}_k}{dt} = -\mathbf{V} + \frac{\partial \phi(\mathbf{l}_k)}{\partial \mathbf{l}_k},$$

is the same as (5.3).

5.2. SMBK and $\mathrm{BMR}_{\Gamma^*=\Gamma}$ brackets

The Hamiltonian structure of the SMBK and $\mathrm{BMR}_{\Gamma^*=\Gamma}$ equations will now be compared.

Hamiltonian functions: For the SMBK equations, the Hamiltonian function H for the case of a circular cylinder is obtained from (3.6) by putting $\Omega = 0$ and noting that M reduces to a 2×2

scalar matrix with entries c. Thus (referring to (2.7)),

$$H = \frac{1}{2}c\langle \mathbf{V}, \mathbf{V}\rangle - \sum_{k,j(k>j)} \Gamma_k\Gamma_j G(\mathbf{l}_k; \mathbf{l}_j) - \frac{1}{2}\sum \Gamma_j^2 g(\mathbf{l}_j; \mathbf{l}_j) \qquad \text{(SMBK)}$$

where G and g are given by (5.9) and (5.8), respectively.

For the $\mathrm{BMR}_{\Gamma^*=\Gamma}$ equations, the Hamiltonian function (equation (1.2) in [5]) is given by

$$H = \frac{1}{4\pi}c\langle \mathbf{V}, \mathbf{V}\rangle + \frac{1}{8\pi^2}\sum \Gamma_j^2 \log(l_j^2 - R^2) - \frac{1}{8\pi^2}\sum \Gamma_j \Gamma \log l_j^2 + \\ + \frac{1}{8\pi^2}\sum_{k,j(k>j)} \Gamma_k\Gamma_j \log \frac{R^4 - 2R^2\langle \mathbf{l}_k, \mathbf{l}_j\rangle + l_k^2 l_j^2}{l_{kj}^2}, \qquad (\mathrm{BMR}_{\Gamma^*=\Gamma})$$

where $l_{kj}^2 = \mid \mathbf{l}_k - \mathbf{l}_j \mid^2$. Using (5.9) and (5.8), it is a straightforward check to observe that the Hamiltonian functions are the same except for a factor of 2π which presumably again has to do with the BMR value of fluid density.

Poisson Brackets: With the equation for $\mathbf{A}$ ignored, the SMBK equations for a circular cylinder are written on the Poisson manifold $P \equiv (\mathbf{L}, \mathbf{l}_1, \ldots\ldots, \mathbf{l}_N)$ equipped with the Poisson bracket obtained from (4.4) after noting that the Lie-Poisson part is trivial. Thus,

$$\{F, G\}_P = \Gamma\left(\frac{\partial F}{\partial L_y}\frac{\partial G}{\partial L_x} - \frac{\partial F}{\partial L_x}\frac{\partial G}{\partial L_y}\right) + \sum_{j=1}^{N}\frac{1}{\Gamma_j}\left\langle \nabla_j F, J^{-1}\nabla_j G\right\rangle,$$

where $F, G : P \to \mathbb{R}$.

The $\mathrm{BMR}_{\Gamma^*=\Gamma}$ equations are written on the Poisson manifold $\tilde{P} = (\mathbf{V}, \mathbf{l}_1, \ldots.., \mathbf{l}_N)$ and equipped with a Poisson bracket whose Poisson tensor elements are given by equation (2.1) in [5].

Consider now the map $\pi : P \to \tilde{P}$ given by the inverse of (5.4) and the identity on the $\mathbf{l}_j$s. In other words,

$$\pi(\mathbf{L}, \mathbf{l}_1, \ldots, \mathbf{l}_N) = (\mathbf{V}, \mathbf{l}_1, \ldots, \mathbf{l}_N) = \\ = \left(\frac{1}{c}\left(\mathbf{L} + \sum \mathbf{k}\times\Gamma_j\mathbf{l}_j + R^2\sum\Gamma_j\left(\frac{x_j}{x_j^2+y_j^2}, \frac{y_j}{x_j^2+y_j^2}\right)\times\mathbf{k}\right), \mathbf{l}_1, \ldots.., \mathbf{l}_N\right) \qquad (5.10)$$

We then obtain the following:

Lemma 5.1. *The map $\pi : P \to \tilde{P}$, given by* (5.10), *is a Poisson map.*

Proof.

For functions, $\tilde{F}, \tilde{G} : \tilde{P} \to \mathbb{R}$, the map above defines functions $\tilde{F}\circ\pi, \tilde{G}\circ\pi : P \to \mathbb{R}$. A straightforward application of the chain rule in calculus then gives:

$$\frac{\partial(\tilde{F}\circ\pi)}{\partial L_x} = \frac{1}{c}\frac{\partial\tilde{F}}{\partial V_x},$$

$$\frac{\partial(\tilde{F}\circ\pi)}{\partial L_y} = \frac{1}{c}\frac{\partial\tilde{F}}{\partial V_y},$$

$$\frac{\partial(\tilde{F}\circ\pi)}{\partial x_1} = \frac{\Gamma_1}{c}\left(\frac{\partial\tilde{F}}{\partial V_x}\left(\frac{-2x_1y_1R^2}{l_1^4}\right) + \frac{\partial\tilde{F}}{\partial V_y}\left(1 + R^2\frac{x_1^2 - y_1^2}{l_1^4}\right)\right) + \frac{\partial\tilde{F}}{\partial x_1},$$

$$\frac{\partial(\tilde{F}\circ\pi)}{\partial y_1} = \frac{\Gamma_1}{c}\left(\frac{\partial\tilde{F}}{\partial V_x}\left(-1 + R^2\frac{x_1^2 - y_1^2}{l_1^4}\right) + \frac{\partial\tilde{F}}{\partial V_y}\left(\frac{2x_1y_1R^2}{l_1^4}\right)\right) + \frac{\partial\tilde{F}}{\partial y_1},$$

$$\cdots = \cdots,$$

$$\cdots = \cdots$$

Therefore,

$$\left\{\tilde F\circ\pi,\tilde G\circ\pi\right\}_P=$$
$$=\left(\frac{\partial\tilde F}{\partial V_y}\frac{\partial\tilde G}{\partial V_x}-\frac{\partial\tilde F}{\partial V_x}\frac{\partial\tilde G}{\partial V_y}\right)\left(\frac{\Gamma}{c^2}+\sum\frac{\Gamma_j}{c^2}\frac{R^4-l_j^4}{l_j^4}\right)+\left(\frac{\partial\tilde F}{\partial V_y}\frac{\partial\tilde G}{\partial x_1}-\frac{\partial\tilde F}{\partial x_1}\frac{\partial\tilde G}{\partial V_y}\right)\left(-\frac{1}{c}\frac{2x_1y_1R^2}{l_1^4}\right)+$$
$$+\left(\frac{\partial\tilde F}{\partial V_y}\frac{\partial\tilde G}{\partial y_1}-\frac{\partial\tilde F}{\partial y_1}\frac{\partial\tilde G}{\partial V_y}\right)\left(\frac{1}{c}\left(1+R^2\frac{x_1^2-y_1^2}{l_1^4}\right)\right)+$$
$$+\left(\frac{\partial\tilde F}{\partial V_x}\frac{\partial\tilde G}{\partial x_1}-\frac{\partial\tilde F}{\partial x_1}\frac{\partial\tilde G}{\partial V_x}\right)\left(\frac{1}{c}\left(1-R^2\frac{x_1^2-y_1^2}{l_1^4}\right)\right)+$$
$$+\left(\frac{\partial\tilde F}{\partial V_x}\frac{\partial\tilde G}{\partial y_1}-\frac{\partial\tilde F}{\partial y_1}\frac{\partial\tilde G}{\partial V_x}\right)\left(-\frac{1}{c}\frac{2x_1y_1R^2}{l_1^4}\right)+\cdots,$$

and it is easily seen, keeping in mind Fact A and noting a factor of 2π, that the above is the same bracket as equation (2.1) in [5]. In other words,

$$\left\{\tilde F\circ\pi,\tilde G\circ\pi\right\}_P=\left\{\tilde F,\tilde G\right\}_{\tilde P}$$
■

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