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Vortex Motion and the Geometric Phase. Part I. Basic Configurations and Asymptotics

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Summary. The *geometric,* or *Hannay-Berry,* phase is calculated for three canonical point vortex configurations in the plane. The simplest configuration is the three-vortex problem with arbitrary (like signed) circulations, where two of the vortices are near each other compared to the distance between them and a third vortex. We show that the third (distant) vortex induces a geometric phase in the relative angle variable between the two nearby vortices. The second configuration is a particle and vortex in a circular domain. In this problem, the geometric phase is induced on the particle by the presence of the boundary. The third configuration is an infinite row of point vortices undergoing subharmonic pairing. In this case, a geometric phase is induced on a particle orbiting one of the vortices as the vortex pairs orbit each other. In each case we derive the formula for the geometric phase using an asymptotic procedure, then we give it a geometric interpretation. For the asymptotic derivation, we show how the geometric phase can be interpreted as the product of two terms, one of which goes to zero, the other to infinity. Because they go at rates that balance each other, there is a residual $O(1)$ term in the limit $\epsilon \to 0$. In this way, we can see that the $\epsilon \to 0$ problem is fundamentally different from the $\epsilon = 0$ problem. For the geometric interpretation, we introduce a 1-form γ defined on the plane and show that the phase θ_g in the appropriate angle variable can be constructed by taking the contour integral of this 1-form over the closed vortex path. In each case this gives the simple formula $\theta_g = \oint \gamma$.

Key words. geometric phase, Berry phase, point vortex, mixing layer

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1. Introduction to Geometric Phases

In 1984, Berry [1] investigated the evolution of a quantum system whose Hamiltonian depends on external parameters which are varied slowly in a closed loop. The adiabatic theorem in quantum mechanics [2] states that, for (infinitely) slow changes, the system wave function, whose evolution is described by the time-dependent Schrödinger equation, is instantaneously in an eigenstate of the Hamiltonian (for the values of the external parameters at that instant). Hence, at the end of the cycle when the parameters return to their original values, the wave function should return to the eigenstate it started in, except for a possible change in phase. This phase factor was long thought of as being only a dynamic phase factor describing the time effect of the cycle. However, Berry showed that this was not the only phase contribution; there also is a geometric part that is given by a circuit integral in parameter space and is, thus, independent of the dynamics along the circuit (provided that it is still slow enough for the adiabatic theorem to hold). This phase factor is now commonly referred to as Berry's phase and depends only on the geometry of the closed loop in parameter space.

Hannay [3] subsequently studied the classical analogue of Berry's phase, now commonly referred to as Hannay's angle, by considering the slow evolution of integrable Hamiltonian systems. For such systems there exist, in principle, a set of canonical variables called action and angle variables [4]. The action variables are invariants of the system and in periodic systems represent the area of the closed trajectory of the system in phase space. The conjugate angle variables evolve linearly in time at the system frequencies. Hannay studied integrable systems whose Hamiltonian function depends on certain external parameters. If these parameters do not change in time the Hamiltonian is time-independent and is a conserved quantity along the solutions of the system. He then analysed 'the fate of the angle variables' in the case when the parameters change slowly in a closed loop. Instantaneously, the system evolves at some frequency that depends on the parameter values at that instant. Intuitively, therefore, one would expect that the total angle change at the end of the cycle is just the time integral of the instantaneous evolution over the period of the cycle. Hannay showed, however, that this need not be true. There could be an extra angle change that, like in the quantum case, depends solely on the circuit in parameter space of the closed loop.

Following this, Berry [5] obtained a semiclassical relationship between his phase and Hannay's angle. Aharonov and Anandan [6] then showed that the geometric phase could be extracted from the total phase change for any general cyclic evolution, not necessarily adiabatic, of a quantum system. Berry and Hannay [7] investigated the classical analogue and found a similar geometric phase for a general nonadiabatic cyclic change. Golin and co-workers wrote a series of papers dealing with different issues related to the Hannay angles in classical Hamiltonian systems—the existence of Hannay angles for smooth systems with one degree-of-freedom [9], Hannay angles in the presence of symmetries [10], and measurement of Hannay angles [11] (see also Golin [12], Golin, Knauf, and Marmi [13], [14]).

In a more geometric vein, Simon [16], commenting on Berry's earlier paper, showed that his phase can be interpreted as the holonomy associated with the connection in a line bundle over the parameter space. The adiabatic evolution provides a path (the connection) along which the wave function's initial eigenvector is transported. Anandan and Stodolsky [19] showed that the interpretation can be extended to the holonomy in a vector bundle by considering all the eigenspaces of the wave function. Marsden, Montgomery, and Ratiu [20], [21], Marsden and Ratiu [23], Montgomery [26], and Golin, Knauf, and Marmi [13] develop these concepts further, especially in the classical case and their extensions to nonintegrable systems. They show that averaging defines a connection which can be related to Ehresmann and Cartan connections on fiber bundles [17], [18] and that the Hannay angle is the holonomy of this connection. Nonintegrable classical systems have also been examined by Robbins and Berry [15]. Levi [27] examines some simple rigid body motions which have geometric phases and their relation to parallel transport.

In other work, Montgomery, examining the rotation of a free rigid body and the gravitational three-body problem [8], [28], has shown how exact formulae for angle changes in these problems can be derived which show the characteristic splitting into a "dynamic phase" and a "geometric phase." Alber and Marsden [24] have shown how the phase shift formula for soliton interactions can be interpreted as a geometric phase. R. Newton [25] derived the Berry phase formula associated with Schrödinger operators with a continuous spectrum and relates it to the well-known S matrix from scattering theory. In a series of papers [47], [49], [50], Shapere and Wilczek have shown how a geometric phase arises in the context of self-propulsion of micro-organisms at very low Reynolds number regimes. This point of view is closely related to recent developments of the geometric phase in the context of control theory, which is discussed in [22]. Finally, Marsden and Scheurle [29] use the geometric phase idea on mechanical systems with symmetries to show how symmetric patterns in the phase space of the system can be brought out that would not otherwise be seen (see also [30]). By now, there are several sources where one can get an overview of the various interpretations and applications of the geometric phase in various contexts. For historical accounts, see [44], [46]. The collection of papers reprinted in [48] gives a nice introduction to the important papers on the subject before 1989, while [54] gives an overview with a focus on quantum and chemical applications.

To understand most simply why the geometric phase, when it arises in the adiabatic limit of a slow change[1] of external parameters, is geometric, it is useful to view it in a general way as follows. Consider the parameter-dependent evolution of some (real) variable $q(t, \mathbf{X}(\epsilon t))$ of any dynamical system, where $\mathbf{X}(\epsilon t)$ denotes the (real) vector of external parameters. Here ϵ measures the rate at which the parameters are varied. The change in the variable value[2] at the end of time $T = 1/\epsilon$ in which one or more of the parameters are varied in a closed loop ($\mathbf{X}(1) = \mathbf{X}(0)$) will be given by

$$\int_{q(0)}^{q(T)} dq(t, \mathbf{X}(\epsilon t)) = \int_0^T \frac{\partial q(t, \mathbf{X}(\epsilon t))}{\partial t}\, dt + \oint \frac{\partial q(t, \mathbf{X})}{\partial \mathbf{X}} \cdot d\mathbf{X}. \tag{1}$$

The first integral measures the cumulative effect of "local" changes due to the instantaneous evolution and is identified as the dynamic phase of the system. The second integral

[1] In the limit of an infinitely slow change.

[2] In most of the classical systems of interest the variable is usually an angle variable and the change is referred to as a phase change.

is over the closed loop in parameter space. If the integrand is bounded for all time then this integral is finite. In general, it will be a function of T and hence ϵ, and may converge to a nonzero limiting value as $\epsilon \to 0$. To see why the contour integral in parameter space, if it converges, can give rise to a nonzero limit, we can rewrite it as a time integral, i.e.,

$$\oint \frac{\partial q(t, \mathbf{X})}{\partial \mathbf{X}} \cdot d\mathbf{X} = \int_0^T \dot{\mathbf{X}}(\epsilon t) \cdot \frac{\partial q(t, \mathbf{X}(\epsilon \mathbf{t}))}{\partial \mathbf{X}(\epsilon \mathbf{t})} dt, \tag{2}$$

where $\dot{\mathbf{X}}(\epsilon t)$ denotes the slow rate at which the parameters are changed. As $\epsilon \to 0$, $\dot{\mathbf{X}}(\epsilon t) \to 0$ and $T \longrightarrow \infty$. If $\frac{\partial q(t,\mathbf{X})}{\partial \mathbf{X}}$ is bounded for all times, then the integrand decreases as the range of integration becomes larger. There is no reason, a priori, to believe that in the limit of vanishing ϵ the integral is zero. These quantities can approach their limits in such a way as to give a nonzero result.

The geometric phase is this nonzero limit of the ϵ-dependent contour integral. In the adiabatic limit, the ϵ dependency is removed and the integral becomes a function solely of the closed loop in parameter space, i.e., a purely geometric quantity. For small ϵ, the integral can thus be viewed as an $O(1)$ term (the geometric phase), plus $O(\epsilon)$ corrections. Assuming the limit and integration process can be interchanged, the geometric nature of the phase is evident even more clearly by replacing the limit of the contour integral in (1) with the contour integral of the integrand as $\epsilon \to 0$. This limit, if it exists, then defines a mapping from the parameter space to the real line. The geometric phase can then be viewed as the contour integral of a 1-form defined on the parameter space.[3]

In most systems of mathematical and engineering interest, the above limiting procedures cannot be used directly for calculating the geometric phase because the variable q typically represents the unknown integral curve of some dynamical system. The problem then is to extract the phase in such systems without knowing exact solutions, or even the action-angle form. The slow evolution of the parameters introduces a second timescale in the problem and this feature can be exploited to calculate the phase. For example, in Hamiltonian systems of the type considered by Hannay, the method of averaging yields the phase in the angle variable. The particular method used by Hannay and Berry, however, requires knowledge of the dependence of the canonical coordinates on the external parameters [3], [5]. In the absence of such information, it seems natural to ask whether one can calculate the geometric phase by constructing an asymptotic series solution in ϵ for the differential equations governing q. Such a solution would also give information on the ϵ dependency of the total phase. We may mention that asymptotic series solutions have been considered before by Bhattacharjee and Sen [52]. Berry [51] himself has suggested an iterative scheme for calculating the phase.

To construct such a solution, one could use either the method of averaging or the multiscale technique [32], [31]. In this paper we develop the latter owing to the relative computational ease of calculating the higher order terms in the series. In this technique, one introduces an independent slow time variable $\tau = \epsilon t$ on which the parameters $\mathbf{X}(\tau)$ vary. We then view the variable $q(t, \tau)$ as a function of these two independent time

[3] A natural extension of this idea (using the generalized Stokes theorem) is to view the geometric phase as the integral of a 2-form defined over the region of the parameter manifold enclosed by the contour.

variables. The formula for the total phase change then reads:

$$\int_{q(0)}^{q(T)} dq(t,\tau) = \int_0^{T=1/\epsilon} \frac{\partial q(t,\tau)}{\partial t} \cdot dt + \int_0^1 \frac{\partial q(t,\tau)}{\partial \tau} \cdot d\tau, \tag{3}$$

where the second integral on the right-hand side is the same as the contour integral in (1). The geometric phase, if it exists, should arise from this integral. The multiscale technique gives an asymptotic series representation for $q(t,\tau)$ in the form

$$q(t,\tau) = q_0(t,\tau) + \epsilon q_1(t,\tau) + \epsilon^2 q_2(t,\tau) + \cdots.$$

This gives an asymptotic series representation for the second integral,

$$\int_0^1 \frac{\partial q}{\partial \tau} \cdot d\tau = \int_0^1 \frac{\partial q_0}{\partial \tau} \cdot d\tau + \epsilon \int_0^1 \frac{\partial q_1}{\partial \tau} \cdot d\tau + \cdots. \tag{4}$$

The $O(1)$ terms in this series are then identified as the geometric phase terms.

This paper is structured in the following way. We start in Section 2 by giving a general formulation for the calculations to be carried out. In 2.1 we describe the basic set-up and highlight the main results for the three canonical configurations to be treated. Since the form of the equations are similar for each configuration, we describe the general form of the equations in Section 2.2 and show how the multiscale method leads to the identification of the geometric phase term. In Section 2.3 we show how this term can be reinterpreted in a geometric way. Sections 3, 4, and 5 contain the details of the calculations for each of the three canonical configurations that we analyse. The problem described in Section 3 appears to be the simplest point vortex configuration giving rise to a nontrivial phase factor. The configuration analysed in Section 4 shows that a geometric phase can be induced by a solid boundary in the flow. The corresponding problem, without boundaries achieved by using image vortices, is related to the three-vortex problem discussed in Section 3, although it cannot be directly derived from it as a special case. The shear layer model described in Section 5 is the most complex of the three configurations. It is related to the first as well and can be thought of as a restricted three-vortex problem in a periodic strip.

2. General Formulation for Geometric Phase Calculations

2.1. Modus Operandi

In this section, we formulate our approach to identifying and computing the "geometric" or "Hannay-Berry" phase in three problems in planar, incompressible, inviscid fluid flows involving point vortices. In each of our problems, we track the position of a "phase object" which for our purposes could be a fluid particle, a passive tracer particle, or a point vortex of arbitrary strength. Typically, we are interested in the limiting case where the phase object is close to another point vortex, which we refer to as the "parent vortex." The phase object moves under the influence of the parent vortex as well as the influence of an additional vortex or vortices placed farther away, which we refer to as the "farfield vortices." Since it is nearby, the parent vortex causes rapid revolution of the phase object

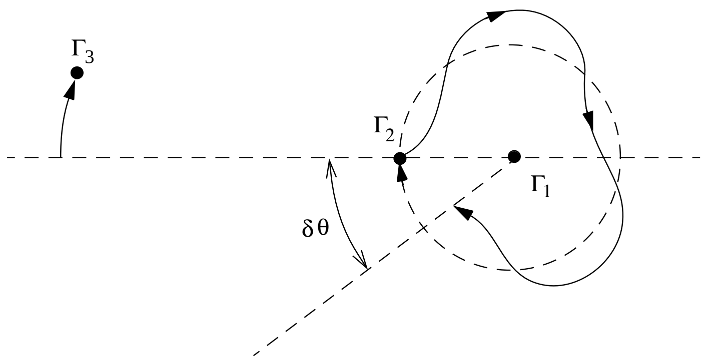

Fig. 1. A schematic sketch showing the effect of the flow field of Γ_3 on the motion of Γ_2. In the absence of Γ_3, the motion of Γ_2 (in a frame translating with Γ_1) is shown by the dashed circle. The presence of Γ_3 distorts this circular orbit in the manner shown by the solid curve and causes an angular perturbation of $\delta\theta$ per time period of Γ_2's unperturbed orbit. These perturbations accumulate over Γ_3's time period to give the geometric phase.

with a time period that we denote as T_s. Since the velocity field of a planar point vortex scales like $1/r$, it is clear that $T_s \sim r^2(0)$, where $r(0)$ denotes the initial distance between the phase object and its parent vortex. The additional dynamics due to the farfield vortices are such that they induce a periodic motion of the vortices,[4] so that we can define a longer period $T_l \sim D^2(0) >> T_s$, where $D(0)$ is the initial distance between the farfield vortex and the parent vortex. We thus have a natural small parameter at our disposal, which we define as $\epsilon^2 = T_s/T_l \sim r^2(0)/D^2(0)$. We then nondimensionalize the problem so that $T_s = O(1)$; hence, $T_l = O(1/\epsilon^2)$. The time-varying periodic coefficients that appear (due to the periodic vortex motion) in the equations of motion of the phase object can then be viewed as varying on the slow timescale $\tau \sim \epsilon^2 t$ and we have an adiabatic process as $\epsilon \to 0$.

The angle change (relative to the parent vortex) of the phase object is then computed at the end of time T_l. In the adiabatic limit, this angle change is shown to split naturally into two parts: a "fast" part and a "slow" part. The fast part, which comes from the dynamic phase, is what would be present if the phase object rotated only around the parent vortex, with no farfield vorticity present. We call this the "$\epsilon = 0$," or "unperturbed" problem. The slow part is the geometric phase θ_g. As described in Section 1, θ_g arises from the limiting adiabatic procedure $\epsilon \to 0$ and can be interpreted in two distinct ways.

First, we describe what can be called an asymptotic interpretation. Here, we view the contribution θ_g as arising from the product of two terms: $\theta_g = \delta\theta \cdot N$. The first term, $\delta\theta$, is defined as the angle difference between the unperturbed and perturbed phase object at any given fixed time t^* (see Figure 1), and hence

$$\delta\theta = \theta_\epsilon(t^*) - \theta_0(t^*).$$

Since the perturbed and unperturbed equations approach each other smoothly as $\epsilon \to 0$,

[4] The periodic motion is induced in the farfield vortices themselves and, in some of the problems, in the parent vortex as well.

we know that $\delta\theta \to 0$ as $\epsilon \to 0$. In all point vortex problems, scaling requires that $\delta\theta \sim C_1\epsilon^2$. The second term, N, is defined to be the number of complete orbits of the phase object during one complete cycle of the farfield vortices. Since the phase object has period T_s and the farfield vortices have period T_l, we have

$$N \sim T_l/T_s \sim C_2/\epsilon^2.$$

The geometric phase contribution then becomes

$$\theta_g = \delta\theta \cdot N = (C_1\epsilon^2) \cdot (C_2/\epsilon^2) = C_1 C_2 \sim O(1).$$

It arises from the **limiting procedure** $\epsilon \to 0$ as the balance between one term going to zero, the other to infinity. Notice that in this interpretation the limiting procedure $\epsilon \to 0$ is distinct from the limit $\epsilon = 0$. Such an asymptotic balance is achieved in all of the vortex problems we treat, and we believe that this kind of balance is present in many other geometric phase problems discussed in the literature.

The second, "geometric" interpretation of θ_g arises from the periodicity in the vortex motion. This defines a closed loop in the physical plane—the path of the periodic vortex.[5] We introduce a 1-form γ [33], [34] on the plane, and show that

$$\theta_g = \oint \gamma,$$

where the contour integral is around the closed loop. In this formulation, θ_g is evidently independent of the time period T_l. This is consistent with the asymptotic interpretation in which the small parameter ϵ ultimately drops out of the formulation. In the remaining sections we analyse three canonical vortex flows where this decomposition occurs. In the first two flows, θ_g is written in a geometric form reminiscent of that given for the 'particle-on-a-hoop' problem [3], [23].

Problem 1: Three point vortices in the plane. This is the simplest vortex configuration in which a geometric phase occurs and is shown in Figure 2. All the point vortex strengths are of the same sign, but can be of arbitrary magnitude $(\Gamma_1, \Gamma_2, \Gamma_3)$. Without loss of generality, we take Γ_1 as the parent vortex,[6] Γ_2 as the phase object, and Γ_3 as the farfield vortex. Our result for the geometric phase induced on the phase object (Γ_2) is

$$\theta_g = \frac{\Gamma_3}{\Gamma_1 + \Gamma_2 + \Gamma_3} \cdot 2\pi \cdot \cos(2\theta_i), \tag{5}$$

where θ_i is the initial condition for the phase object.

Problem 2: A point vortex and particle in a circle. In this problem, we show that a geometric phase contribution arises as a result of a boundary effect. Here, the phase object is a passive particle orbiting the parent vortex Γ as shown in Figure 3. The parent vortex

[5] This could be either the farfield vortex or the parent vortex.

[6] Here and elsewhere we refer to vortices by their strengths. We assume positive strengths everywhere. The effect of changing the sign of the strengths in each problem is merely to change the sign of the geometric phase.

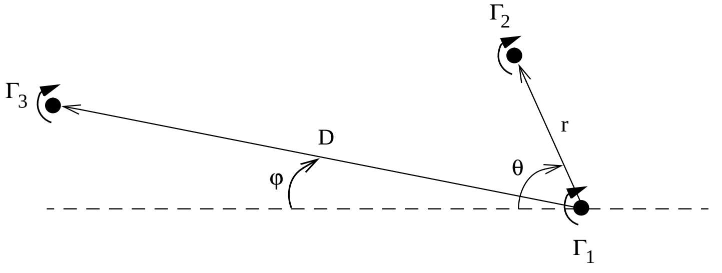

Fig. 2. Three point vortices (filled circles) of positive strengths in an unbounded plane. The geometric phase is calculated for the variable θ.

in any eccentric position moves in a closed circular path with radius R_1 and with constant frequency. The farfield vorticity is due to the circular boundary of radius $R_2 > R_1$. Equivalently, we can think of the farfield vortex as an image vortex $-\Gamma$ placed at its image point R_2^2/R_1 outside the circle [59], [60]. The geometric phase contribution on the phase object is

$$\theta_g = -\frac{2\pi}{(R_2/R_1)^2 - 1} \cdot \cos(2\theta_i). \tag{6}$$

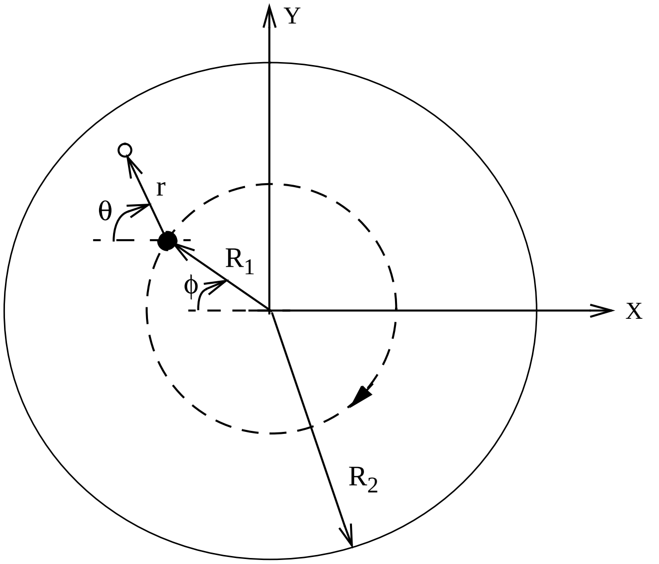

Fig. 3. A point vortex (filled circle) and a fluid particle (unfilled circle) in a circular domain. Orbit of the vortex is shown by the dashed circle. The cartesian frame $X - Y$ is centered at the center of the circular domain.

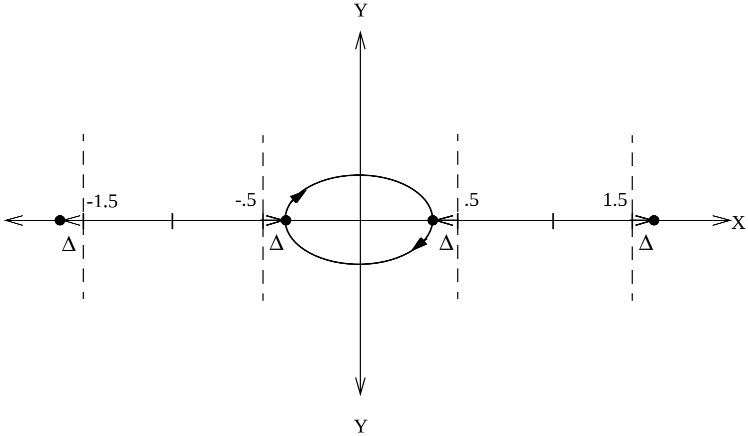

Fig. 4. The positions of the vortices (filled circles) immediately after the subharmonic perturbations (of magnitude Δ and direction shown by arrows along X-axis). The subsequent motion of the vortices is shown in the central window. This motion is identical in every other window. The windows are divided by the vertical dashed lines which mark the initial positions of the vortices (here, $a = 1$).

Note it is independent of the vortex strength.

Problem 3: Pairing in an infinite row of point vortices. Here, we show that a geometric phase contribution arises during the "vortex pairing" stage of nonlinear shear layer evolution in a simple two-dimensional model for the process thought to be fundamental for the generation of small-scale motion and enhanced mixing in a wide range of more complicated real flows [35], [36]. In the model, an infinite row of evenly spaced, equal strength vortices is given a subharmonic perturbation so that neighboring vortices pair up and undergo periodic motion as shown in Figure 4. The phase object is a tracer particle near any parent vortex, while the farfield flow is due to the infinite number of other point vortices periodically spaced. Our result for this configuration is

$$\theta_g = \left(\frac{2k+6}{3}\right) \cdot K \cdot \cos(2\theta_i). \tag{7}$$

Here K is the complete Jacobi elliptic integral of the first kind with modulus k [37] arising from the exact solution of the closed vortex orbit.

2.2. Asymptotic Procedure

The three problems we treat, when prepared appropriately, all have the same general form. In this section we outline our asymptotic procedure, based on multiscale theory [32], for computing the geometric phase θ_g.

Let (r, θ) denote the nondimensional polar coordinates of the phase object with respect to the parent vortex. The form of the general equations of motion we encounter are

$$\frac{dr}{dt} = \epsilon^2 f\left(r, \theta, D(\epsilon^2 t), \phi(\epsilon^2 t), \epsilon\right), \tag{8}$$

$$\frac{d\theta}{dt} = \frac{\Omega}{r^2} + \epsilon^2 g\left(r, \theta, D(\epsilon^2 t), \phi(\epsilon^2 t), \epsilon\right). \tag{9}$$

Here, ϵ is a small dimensionless parameter, f and g are the components of the vector field due to the farfield vortices, and D and ϕ are nondimensional polar variables representing the periodic vortex motion. D is the distance between the parent vortex and the farfield vortex and ϕ is the angle with the horizontal axis that the straight line joining these two vortices makes. These variables provide the periodic time varying coefficients in the above equations, with the angle variable ϕ occuring in the argument of some periodic function,

$$D(T_l) - D(0) = 0,$$

$$\phi(T_l) - \phi(0) = 2\pi.$$

Notice that in the limit $\epsilon = 0$, the equations reduce to those governing a passive particle around an isolated point vortex in an unbounded plane.

Since $T_l = O(1/\epsilon^2)$, there exists a slow timescale $\tau = \epsilon^2 t$ in the superimposed field. By the usual multiscale "ansatz," the two timescales (t, τ) are viewed as independent variables. The ordinary differential equations then split into partial differential equations as $\frac{d}{dt} \to \frac{\partial}{\partial t} + \epsilon^2 \frac{\partial}{\partial \tau}$,

$$\frac{\partial r}{\partial t} + \epsilon^2 \frac{\partial r}{\partial \tau} = \epsilon^2 f\left(r, \theta, \tilde{D}, \tilde{\phi}, \epsilon\right), \tag{10}$$

$$\frac{\partial \theta}{\partial t} + \epsilon^2 \frac{\partial \theta}{\partial \tau} = \frac{\Omega}{r^2} + \epsilon^2 g\left(r, \theta, \tilde{D}, \tilde{\phi}, \epsilon\right), \tag{11}$$

where the $\sim$ overhead denotes dependency on the slow time alone.[7] In order to calculate the geometric phase for such a system, therefore, we seek a two timescale series solution asymptotic in ϵ for given initial conditions $r(0) = 1$ and $\theta(0) = \theta_i$. Thus,

$$r(t, \tau) = r_0(t, \tau) + \epsilon r_1(t, \tau) + \epsilon^2 r_2(t, \tau) + \cdots,$$

$$\theta(t, \tau) = \theta_0(t, \tau) + \epsilon \theta_1(t, \tau) + \epsilon^2 \theta_2(t, \tau) + \cdots,$$

with $r_0(0) = 1$, $\theta_0(0) = \theta_i$, and $r_j(0) = \theta_j(0) = 0$ for $j = 1, 2, \ldots$. Substituting in (10) and (11), we Taylor expand the functions f, g, and $1/r^2$ about $\epsilon = 0$, assuming they possess ϵ-derivatives of all orders in a neighborhood of that point. Thus, for example,

$$f\left(r, \theta, \tilde{D}, \tilde{\phi}, \epsilon\right) = f_0 + \epsilon \left(\frac{df}{d\epsilon}\right)_{\epsilon=0} + \frac{\epsilon^2}{2}\left(\frac{d^2 f}{d\epsilon^2}\right)_{\epsilon=0} + \cdots,$$

[7] This notation is used throughout the paper.

where $f_0 = f(r_0, \theta_0, \tilde{D}, \tilde{\phi}, 0)$, and

$$\left(\frac{df}{d\epsilon}\right)_{\epsilon=0} = \left(\frac{\partial f}{\partial \epsilon}\right)_{\epsilon=0} + \left(\frac{\partial f}{\partial r}\frac{dr}{d\epsilon}\right)_{\epsilon=0} + \left(\frac{\partial f}{\partial \theta}\frac{d\theta}{d\epsilon}\right)_{\epsilon=0},$$
$$= \left(\frac{\partial f}{\partial \epsilon}\right)_{\epsilon=0} + r_1\left(\frac{\partial f}{\partial r}\right)_{\epsilon=0} + \theta_1\left(\frac{\partial f}{\partial \theta}\right)_{\epsilon=0},$$

and the higher derivatives in the Taylor expansion are similarly computed. Note that at $\epsilon = 0$, $r = r_0$ and $\theta = \theta_0$. Equating like powers of ϵ gives a pair of first order PDEs at every order and these are solved sequentially from $O(1)$. The solutions contain arbitrary functions of slow time which are determined uniquely by identifying terms that can cause growth on the fast time at higher orders and eliminating them, i.e., imposing the so-called "solvability conditions."

Consider the first three orders in ϵ:

$$O(1): \frac{\partial r_0}{\partial t} = 0,$$
$$\frac{\partial \theta_0}{\partial t} = \frac{\Omega}{\tilde{r}_0^2},$$
$$O(\epsilon): \frac{\partial r_1}{\partial t} = 0,$$
$$\frac{\partial \theta_1}{\partial t} = -\frac{2\Omega\tilde{r}_1}{\tilde{r}_0^3},$$
$$O(\epsilon^2): \frac{\partial r_2}{\partial t} = -\frac{d\tilde{r}_0}{d\tau} + f_0,$$
$$\frac{\partial \theta_2}{\partial t} = -\frac{\partial \theta_0}{\partial \tau} + \frac{\Omega}{r_0^2}\left(\frac{3r_1^2}{r_0^2} - \frac{2r_2}{r_0}\right) + g_0.$$

Solving the $O(1)$ equation gives:

$$r_0 = \tilde{r}_0(\tau), \qquad \tilde{r}_0(0) = 1,$$
$$\theta_0 = \theta_F + \theta_S = \Omega t/\tilde{r}_0^2 + \tilde{\theta}_0(\tau),$$

where $\tilde{\theta}_0(0) = \theta_i$, with $0 \le \theta_i \le 2\pi$. Notice:

1. At leading order there is a decomposition of the angle variable into a fast term, $\theta_F = \Omega t/\tilde{r}_0^2$, and a slow term, $\theta_S = \tilde{\theta}_0(\tau)$.
2. It turns out, as will be shown below, that $\tilde{r}_0 =$ const, and hence the fast term θ_F is the exact angle variable evolution in the $\epsilon = 0$ problem, i.e., where the phase object moves about an isolated point vortex with no external slow field. Evaluated at the end of period T_l this gives, as noted earlier, the contribution of the dynamic phase at this order to the angle change.
3. The slow term $\theta_S = \tilde{\theta}_0(\tau)$ evaluated at the end of period T_l gives rise to the geometric phase, θ_g. It is not present in the $\epsilon = 0$ problem. Our goal is to evaluate $\tilde{\theta}_0(\tau)$ by

computing its solvability condition at higher order. We call the equation for $\tilde{\theta}_0$ the "slow phase" equation.

Proceeding to the next order, we have

$$
\begin{aligned}
O(\epsilon) : r_1 &= \tilde{r}_1(\tau), \\
\tilde{r}_1(0) &= 0, \\
\frac{\partial \theta_1}{\partial t} &= -2\Omega \tilde{r}_1/\tilde{r}_0^3.
\end{aligned}
$$

In order that θ_1 remain bounded on the fast timescale, we impose the solvability condition: $\tilde{r}_1 = 0$. This then gives $\theta_1 = \tilde{\theta}_1(\tau)$ with $\tilde{\theta}_1(0) = 0$. Then at $O(\epsilon^2)$ we get

$$
\begin{aligned}
\frac{\partial r_2}{\partial t} &= -\frac{d\tilde{r}_0}{d\tau} + f_0, \\
\frac{\partial \theta_2}{\partial t} &= -\frac{\partial \theta_0}{\partial \tau} + g_0 - \frac{2\Omega r_2}{\tilde{r}_0^3}.
\end{aligned}
$$

Assuming that the terms f_0 and g_0 do not lead to secular growth (which we will verify in our examples), the solvability condition for $\tilde{r}_0$ is $\frac{d\tilde{r}_0}{d\tau} = 0 \Rightarrow \tilde{r}_0(\tau) = \tilde{r}_0(0) = 1$, and therefore we can solve for r_2 and θ_2,

$$
r_2 = \int f_0 dt + \tilde{r}_2(\tau), \qquad \tilde{r}_2(0) = -\int f_0 \, dt_{|t=0},
$$

and $\theta_2 = \int \left(g_0 - 2\Omega \int f_0 dt\right) dt + \tilde{\theta}_2(\tau)$. The functions f_0 and g_0 now read $f_0 = f(1, \Omega t + \tilde{\theta}_0, \tilde{D}, \tilde{\phi}, 0)$ and $g_0 = g(1, \Omega t + \tilde{\theta}_0, \tilde{D}, \tilde{\phi}, 0)$. The solvability condition for $\tilde{\theta}_0$ then gives the "slow phase" equation,

$$
\boxed{\frac{d\tilde{\theta}_0}{d\tau} = -2\Omega \tilde{r}_2,}
$$

which is fundamental in deriving our formula for the Hannay-Berry phase. Notice at this stage:

1. To solve the phase equation for $\tilde{\theta}_0$, we need to derive the governing equation for $\tilde{r}_2$ by imposing the solvability condition at $O(\epsilon^4)$. Therefore, to derive the "slow phase" system for $(\tilde{\theta}_0, \tilde{r}_2)$, we need to go to $O(\epsilon^4)$. This also shows that the leading order phase implicitly depends on higher order amplitude contributions.
2. In all of the examples we treat, it turns out that the equation governing $\tilde{r}_2$ is

$$
\frac{d\tilde{r}_2}{d\tau} = 0 \qquad \Rightarrow \tilde{r}_2(\tau) = \tilde{r}_2(0) = -\int f_0 \, dt_{|t=0}.
$$

3. With the above assumptions, the solution for the slow phase is

$$\theta_S \equiv \tilde{\theta}_0(\tau) = 2\Omega\tau \int f_0 dt_{|t=0} + \theta_i, \tag{12}$$

$$= 2\Omega\epsilon^2 t \left(\int f_0 \, dt_{|t=0} \right) + \theta_i.$$

We now have the solutions (r, θ) through $O(\epsilon^2)$,

$$r(t,\tau) = 1 + \epsilon^2 r_2(t,\tau) + \cdots,$$

$$\theta(t,\tau) = \Omega t + 2\Omega\tau \int f_0 \, dt_{|t=0} + \theta_i + \epsilon\tilde{\theta}_1(\tau) + \epsilon^2\theta_2(t,\tau) + \cdots, \tag{13}$$

where $r_2(t,\tau)$ and $\theta_2(t,\tau)$ are as derived earlier.

The Hannay-Berry phase can now be calculated by forming the asymptotic series representation of the integral (4) using (13). It takes the form

$$\int_0^\beta \frac{\partial\theta}{\partial\tau} d\tau = \int_0^\beta \frac{\partial}{\partial\tau} \left(\Omega t + 2\Omega\tau \int f_0 dt_{|t=0} + \theta_i \right) \cdot d\tau$$

$$+ \epsilon \int_0^\beta \frac{d\tilde{\theta}_1}{d\tau} \cdot d\tau + \epsilon^2 \int_0^\beta \frac{\partial\theta_2}{\partial\tau} \cdot d\tau + \cdots,$$

where we have assumed the long period $T_l = \beta/\epsilon^2$. In the limit $\epsilon \to 0$ only the first term of the series remains. Due to the decomposition of θ_0 into a fast term and a slow term, we get the Hannay-Berry phase as

$$\theta_g = 2\Omega\beta \int f_0 dt_{|t=0}, \tag{14}$$

$$= \int_0^\beta \frac{d\theta_S}{d\tau} d\tau. \tag{15}$$

This is the general formula for the Hannay-Berry phase. An important observation to make about the final formula for θ_g is that *it does not depend on the function g in equation (9).*

2.3. Geometric Interpretation

Our geometric interpretation of the Hannay-Berry phase is based on the discussion in the paragraph following (2). In the previous section we derived the phase as a time integral (15). In this section we transform it back into a contour integral defined on the parameter space. We identify the physical plane on which the point vortices move as the parameter space in these problems.[8] Briefly, we show that there exists a 1-form defined

[8] Note that this plane is also the phase space for the vortices and for the fluid/tracer particle.

on this plane whose integral around the closed path of the periodic vortex is equal to the Hannay-Berry phase.

We choose a suitable point in the plane as the origin of a Cartesian, fixed X-Y frame.[9] The vortex orbit lies in some deleted neighbourhood U of this origin and is determined uniquely by its initial position x_0 (which we choose arbitrarily to lie on the negative X-axis). We view the orbit as being parametrised by the nondimensional slow time τ. We seek a relation of the form $\tau(X, Y)$ where X, Y are the (dimensional) coordinates of any point in U. This point lies on some vortex orbit corresponding uniquely to some x_0. Once we obtain such a relation, the time integral in (15) can be transformed trivially into a contour integral on U using the linear relation (12),

$$\int_0^\beta \frac{d\theta_S}{d\tau}\, d\tau = \left(2\Omega \int f_0\, dt_{|t=0}\right) \oint \frac{\partial\tau(X, Y)}{\partial X}\, dX + \frac{\partial\tau(X, Y)}{\partial Y}\, dY,$$

where the contour integration is performed over the particular vortex orbit of period $T_l = \beta/\epsilon^2$. This defines a 1-form γ on U given by

$$\gamma = 2\Omega \int f_0\, dt_{|t=0} \left[\frac{\partial\tau(X, Y)}{\partial X}\, dX + \frac{\partial\tau(X, Y)}{\partial Y}\, dY\right],$$

and by the method of its construction we have

$$\theta_g = \oint \gamma.$$

Note that if the Hannay-Berry phase for the particular problem shows a dependency on the vortex orbit, then x_0 will appear in the relation $\tau(X, Y)$ and in the 1-form γ.

A simple way of achieving this in practice is to make use of the (possibly orbit-dependent) relation $\phi_{x_0}(\tau)$, if known exactly.[10] $\phi_{x_0}(\tau)$ is the angle ϕ with the negative X-axis (measured in the clockwise direction) that the vortex makes at time τ in the unique orbit from initial position x_0. For any point with (dimensional) coordinates (X, Y) in U, $\phi = \tan^{-1}(-Y/X)$. Assuming that the one-to-one function $\phi_{x_0}(\tau)$ can be inverted, we can then form the composite map: $\tau = \phi_{x_0}^{-1} \circ \phi$. In terms of the Cartesian coordinates, this is

$$\tau(X, Y) = \phi_{x_0}^{-1} \circ \tan^{-1}(-Y/X) \equiv G\left(\tan^{-1}(-Y/X)\right), \tag{16}$$

and it explicitly defines the relation $\tau(X, Y)$. The time integral is then transformed into a contour integral as

$$\int_0^\beta \frac{d\theta_S}{d\tau}\, d\tau = \left(2\Omega \int f_0\, dt_{|t=0}\right) \oint G' \cdot \left(\frac{Y}{X^2 + Y^2}\, dX - \frac{X}{X^2 + Y^2}\, dY\right). \tag{17}$$

To further simplify the above formula requires explicit knowledge of G'. In our first two problems, $\phi_{x_0}(\tau)$ takes on the simple linear form $\phi_{x_0}(\tau) = (2\pi/\beta)\tau$, where β

[9] The choice of this origin will be made clear in each problem.

[10] The subscript denotes the orbit dependency.

is the constant appearing in the formula for the long time period $T_l = \beta/\epsilon^2$. Hence, $G = (\beta/2\pi)\tan^{-1}(-Y/X)$ and we get $G' = \beta/2\pi$. The above transformation then takes on the more explicit form,

$$\int_0^\beta \frac{d\theta_S}{d\tau}\,d\tau = \left(\frac{\Omega\beta \int f_0\,dt_{|t=0}}{\pi}\right) \oint \left(\frac{Y}{X^2+Y^2}\,dX - \frac{X}{X^2+Y^2}\,dY\right).$$

It is trivial to check that evaluating the above contour integral gives back the geometric phase defined by (14). The 1-form is then given by

$$\gamma = \frac{\Omega\beta \int f_0\,dt_{|t=0}}{\pi} \left(\frac{Y}{X^2+Y^2}\,dX - \frac{X}{X^2+Y^2}\,dY\right). \tag{18}$$

In the last example, such an explicit expression for G' is used only in a limiting case.

3. A Three-Vortex Problem

3.1. Asymptotic Derivation

In this problem there are three point vortices of the same sign in an unbounded plane,[11] two of them close to each other (Γ_1 and Γ_2) and the third (Γ_3) farther away as shown in Figure 2. We analyse the angle holonomy for Γ_2 (the phase object) as it moves primarily under the influence of the field of Γ_1 (the parent vortex), with Γ_3 (the farfield vortex) providing the superimposed field.

The equations governing the point vortex motion can be written compactly in complex form as [39], [38],

$$\dot{z}^*_\alpha = (2\pi i)^{-1} \sum_{\beta=1}^{3} \Gamma_\beta (z_\alpha - z_\beta)^{-1},$$

where $z_\alpha \equiv x_\alpha + i y_\alpha$, $\alpha = 1, 2, 3$, and x_α, y_α are the Cartesian coordinates of the vortices. These equations can be written in real form as

$$\Gamma_\alpha \dot{x}_\alpha = \frac{\partial \mathcal{H}}{\partial y_\alpha}, \tag{19}$$

$$\Gamma_\alpha \dot{y}_\alpha = -\frac{\partial \mathcal{H}}{\partial x_\alpha}, \tag{20}$$

with Hamiltonian $\mathcal{H} \equiv -\frac{1}{4\pi} \sum_{\alpha<\beta}^{N} \Gamma_\alpha \Gamma_\beta \log|z_\alpha - z_\beta|$. For more on point vortex motions, refer to [40], [41], [42], [43].

For calculating the phase, we use intervortex distances and angles as the vortex variables, as shown in Figure 2. r and D denote the distances of Γ_2 and Γ_3 from Γ_1, respectively, and θ and ϕ denote the angles that the lines joining Γ_2 and Γ_3 to Γ_1,

[11] We consider like-signed vortices to prevent the possibility of unbounded motion; see [55].

respectively, make with the horizontal axis. The angles are measured clockwise from the negative X-axis, and clockwise circulations are assumed to have positive signs.[12] This change of variables from the Cartesian coordinates of the original three degree-of-freedom Hamiltonian system (equations (19) and (20)) is of the form[13]

$$y_2 - y_1 = \hat{r}\sin(\theta), \qquad x_2 - x_1 = -\hat{r}\cos(\theta),$$

$$y_3 - y_1 = \hat{D}\sin(\phi), \qquad x_3 - x_1 = -\hat{D}\cos(\phi),$$

$$y_3 - y_2 = \hat{D}\sin(\phi) - \hat{r}\sin(\theta), \qquad x_3 - x_2 = -\hat{D}\cos(\phi) + \hat{r}\cos(\theta).$$

The resulting equations are then nondimensionalised as follows:

$$r = \frac{\hat{r}}{R_i}, \qquad D = \frac{\hat{D}}{D_i}, \qquad t = \omega\hat{t}.$$

Here, R_i and D_i are the initial distances of Γ_2 and Γ_3, respectively, from Γ_1. ω is taken as being proportional to the frequency of Γ_1 and Γ_2 orbiting each other in the absence of Γ_3, i.e., to $1/R_i^2$. The ratio of the initial distances is the perturbation parameter $\epsilon = \frac{R_i}{D_i}$.

The system of nondimensional equations for r, θ, D, ϕ is

$$\frac{dr}{dt} = \frac{-\alpha_3\epsilon}{2\pi D}\sin(\phi-\theta)\left[\frac{1}{1 - \frac{2\epsilon r}{D}\cos(\phi-\theta) + \frac{\epsilon^2 r^2}{D^2}} - 1\right], \tag{21}$$

$$\frac{d\theta}{dt} = \frac{\alpha_1+\alpha_2}{2\pi r^2} + \frac{\alpha_3\epsilon}{2\pi r D}\cos(\phi-\theta)\left[1 - \frac{1}{1 - \frac{2\epsilon r}{D}\cos(\phi-\theta) + \frac{\epsilon^2 r^2}{D^2}}\right] + \frac{\alpha_3\epsilon^2}{2\pi D^2}\left[\frac{1}{1 - \frac{2\epsilon r}{D}\cos(\phi-\theta) + \frac{\epsilon^2 r^2}{D^2}}\right], \tag{22}$$

$$\frac{dD}{dt} = \frac{\alpha_2\epsilon}{2\pi r}\sin(\phi-\theta)\left[1 - \frac{\frac{\epsilon^2 r^2}{D^2}}{1 - \frac{2\epsilon r}{D}\cos(\phi-\theta) + \frac{\epsilon^2 r^2}{D^2}}\right], \tag{23}$$

$$\frac{d\phi}{dt} = \frac{(\alpha_1+\alpha_3)\epsilon^2}{2\pi D^2} + \frac{\alpha_2\epsilon}{2\pi r D}\cos(\phi-\theta)\left[1 - \frac{\frac{\epsilon^2 r^2}{D^2}}{1 - \frac{2\epsilon r}{D}\cos(\phi-\theta) + \frac{\epsilon^2 r^2}{D^2}}\right] + \frac{\alpha_2\epsilon^2}{2\pi D^2}\left[\frac{1}{1 - \frac{2\epsilon r}{D}\cos(\phi-\theta) + \frac{\epsilon^2 r^2}{D^2}}\right], \tag{24}$$

with initial conditions $r(0) = 1, \theta(0) = \theta_i, D(0) = 1, \phi(0) = 0$, and where $\alpha_k = \Gamma_k/(R_i^2\omega)$ are the nondimensionalised vortex strengths.

[12] We follow these conventions in all our problems.

[13] Hats denote dimensional variables that will be nondimensionalised.

We make here several remarks about this system of equations:

1. The equations for (D, ϕ) are coupled to those for (r, θ). In the general formulation described in Section 2.2, we have made the simplifying assumption that (D, ϕ) are known and thus show up as coefficients in the (r, θ) equations. Here, we derive their evolution in time in an asymptotic form simultaneously from the above equations along with r and θ.
2. For $\epsilon = 0$, the equations reduce to a system in which the farfield vortex is infinitely far away and the two vortices Γ_1 and Γ_2 rotate around their center of vorticity.
3. When $0 < \epsilon \ll 1$, the farfield vortex Γ_3 is almost equidistant from the other two. The motion of Γ_1 and Γ_2 can be thought of as a small perturbation of the $\epsilon = 0$ problem; see Figure 1. As discussed later, the motion of Γ_3 approaches the motion it would have if it were co-orbiting a vortex of strength $\Gamma_1 + \Gamma_2$.
4. We define a (dimensional) time period as the time taken by Γ_3 for an angular change of 2π in ϕ. As a consequence of the previous remark, it is easy to see that for small ϵ this time period varies as $\frac{1}{D_i^2}$. Hence, the nondimensionalised time period T is $O(\frac{1}{\epsilon^2})$ and we can assume the slow timescale $\tau = \epsilon^2 t$
5. Note that with $\alpha_2 = 0$ the last two equations decouple from the rest, D is a constant, and ϕ varies linearly with τ. This solution gives the familiar result of two point vortices rotating uniformly about a common (fixed) point. The first two equations then represent the motion of a fluid particle in such a flow (a "restricted" three-vortex problem) and are of the form represented by (8) and (9).

Using the multiscale "ansatz," we get a system of four PDEs (with the above initial conditions), and we seek an asymptotic series solution to this system in powers of ϵ,

$$r = \sum_{j=0}^{\infty} \epsilon^j r_j(t, \tau),$$

$$\theta = \sum_{j=0}^{\infty} \epsilon^j \theta_j(t, \tau),$$

$$D = \sum_{j=0}^{\infty} \epsilon^j D_j(t, \tau),$$

$$\phi = \sum_{j=0}^{\infty} \epsilon^j \phi_j(t, \tau),$$

where

$$r_0(0) = 1, \qquad \theta_0(0) = \theta_i, \qquad D_0(0) = 1, \qquad \phi_0(0) = 0,$$

and $r_j(0) = 0$, $\theta_j(0) = 0$, $D_j(0) = 0$, $\phi_j(0) = 0$, for all other j.

Following the method outlined in Section 2.2 we arrive at the solutions until $O(\epsilon^2)$ as listed below:

$$\begin{aligned}
r_0 &= 1,\\
\theta_0 &= \frac{(\alpha_1+\alpha_2)}{2\pi}t + \tilde{\theta}_0(\tau),\\
D_0 &= 1,\\
\phi_0 &= \frac{(\alpha_1+\alpha_2+\alpha_3)}{2\pi}\tau,
\end{aligned}$$

$$\begin{aligned}
r_1 &= 0,\\
\theta_1 &= \tilde{\theta}_1(\tau),\\
D_1 &= \frac{\alpha_2}{\alpha_1+\alpha_2}\cos(\phi_0-\theta_0) + \tilde{D}_1(\tau),\\
\phi_1 &= -\frac{\alpha_2}{\alpha_1+\alpha_2}\sin(\phi_0-\theta_0) + \tilde{\phi}_1(\tau),
\end{aligned}$$

$$\begin{aligned}
r_2 &= \frac{\alpha_3}{2\,(\alpha_1+\alpha_2)}\cos[2(\phi_0-\theta_0)] + \tilde{r}_2(\tau),\\
\theta_2 &= \frac{\alpha_3}{\alpha_1+\alpha_2}\sin[2(\phi_0-\theta_0)] + \tilde{\theta}_2(\tau),\\
D_2 &= -\frac{\alpha_2}{(\alpha_1+\alpha_2)}(\tilde{\phi}_1-\tilde{\theta}_1)\sin(\phi_0-\theta_0)\\
&\quad -\left[\frac{\alpha_2}{2(\alpha_1+\alpha_2)}\right]^2\cos(2(\phi_0-\theta_0)) + \tilde{D}_2(\tau),\\
\phi_2 &= \left[\frac{\alpha_2}{(\alpha_1+\alpha_2)}\right]^2\frac{\sin(2(\phi_0-\theta_0))}{2} + \frac{\alpha_2}{(\alpha_1+\alpha_2)}\tilde{D}_1\sin(\phi_0-\theta_0)\\
&\quad -\frac{\alpha_2}{(\alpha_1+\alpha_2)}(\tilde{\phi}_1-\tilde{\theta}_1)\cos(\phi_0-\theta_0) + \tilde{\phi}_2(\tau).
\end{aligned}$$

The solution for θ_0 displays the characteristic decomposition into a fast term and a slow term. The slow phase equation for $\tilde{\theta}_0(\tau)(\equiv\theta_S)$ comes from imposing the solvability condition in the θ_2 equation. Thus,

$$\frac{d\tilde{\theta}_0}{d\tau} = -\frac{(\alpha_1+\alpha_2)}{\pi}\tilde{r}_2, \qquad \tilde{\theta}_0(0) = \theta_i.$$

The solutions listed above give all the information necessary[14] to solve for $\tilde{r}_2(\tau)$ by imposing the solvability condition in the $O(\epsilon^4)$ equation in r. Before doing that, however, we make the following important point.

The leading order terms of D and ϕ do not depend on the fast time. It is clear looking at D_0 and ϕ_0 that Γ_3, to leading order, moves as if co-orbiting a vortex of strength $\Gamma_1+\Gamma_2$. It is only this circular motion of the farfield vortex that is relevant to the geometric phase. As is shown below, the phase is determined by the time period of this two-vortex motion.

[14] Note that the explicit form of the other slow functions, for example $\tilde{\theta}_1$, $\tilde{\phi}_1$, etc., need not be known.

The higher order terms in D and ϕ (which do depend on the fast time) do not play a role. Indeed, if one were to solve (21) and (22) with $D = D_0$, $\phi = \phi_0$ for the time period of this two-vortex motion, then one would get the same solutions for r_i, θ_i $(i = 0, 1, 2)$ leading to the same value for the geometric phase. Making the further observation that (21) and (22) represent, in such a case, the nondimensional equations of motion for a fluid particle in the flow field of two vortices of strengths $\Gamma_1 + \Gamma_2$ and Γ_3 leads us to the following:

Proposition. *The geometric phase in this three-vortex problem is the same as the geometric phase in a 'restricted' three-vortex problem obtained by replacing* Γ_1 *with* $\Gamma_1+\Gamma_2$ *and* Γ_2 *with a fluid particle/passive tracer particle.*

Proceeding with the computations, the equation we finally get for $\tilde{r}_2(\tau)$ is

$$\frac{d\tilde{r}_2}{d\tau} = 0, \qquad \tilde{r}_2(0) = -\frac{\alpha_3}{2(\alpha_1+\alpha_2)}\cos(2\theta_i).$$

Hence, the slow phase for this problem is

$$\theta_S \equiv \tilde{\theta}_0 = \frac{\alpha_3\tau}{2\pi}\cos(2\theta_i) + \theta_i. \tag{25}$$

Next, we estimate the long time period T as defined in Remark 4 of this section. Since this is not known exactly, we derive it as an asymptotic series using the asymptotic series solution for ϕ obtained from the above analysis. We are interested in only the leading term[15] for T. Hence, assuming $T \sim T_0/\epsilon^2$ (T_0 is a constant), we do a leading order balance in the equation $2\pi = \phi_0(T_0/\epsilon^2) + \epsilon\phi_1(T_0/\epsilon^2) + \cdots$. This gives

$$T \sim \frac{4\pi^2}{(\alpha_1+\alpha_2+\alpha_3)\epsilon^2},$$

which is the time period of the two-vortex motion mentioned above. The Hannay-Berry phase is then obtained from (15). Thus,

$$\theta_g = \int_0^{T_0}\frac{d\theta_S}{d\tau}d\tau = \left(\frac{\Gamma_3}{\Gamma_1+\Gamma_2+\Gamma_3}\right)2\pi\cos(2\theta_i). \tag{26}$$

We can consider some special cases of the above formula:

Case (i). If $\Gamma_3 = 0$, then there exists only one timescale in the problem, there is no slowly varying background flow, and hence, there is no geometric phase.

Case (ii). If $\Gamma_1 = \Gamma_2 = \Gamma_3$, then one gets the phase for vortex 2 as

$$\theta_g = \frac{2\pi}{3}\cos 2\theta_i.$$

---

[15] Higher order terms in T do not contribute to the phase.

Case (iii). If $\Gamma_2 = 0$, then one gets the phase for a fluid particle in the flow field of Γ_1 and Γ_3 as

$$\theta_g = \frac{\Gamma_3}{\Gamma_1 + \Gamma_3} 2\pi \cos 2\theta_i. \tag{27}$$

The special case where $\Gamma_3 = \Gamma_1$ was analysed by Newton [53].

3.2. Geometric Interpretation

It is clear from the observations made in the last section that there exists a well-defined closed path (a circle) in the plane associated with the geometric phase. This closed path is that of Γ_3 co-orbiting $\Gamma_1 + \Gamma_2$ in a two-vortex motion. This motion is determined by the variables $D(\tau) = D_0 = 1$ and $\phi(\tau) = \phi_0 = (\alpha_1 + \alpha_2 + \alpha_3)\tau/2\pi$. The same geometric phase is thus obtained by solving (21) and (22) in which these variables appear as periodic coefficients varying on the slow time alone. These equations are of the general form (8) and (9). To obtain the 1-form on the plane whose contour integral along the above path gives us the geometric phase, we follow the method outlined in Section 2.3.

Consider the center of vorticity of the two-vortex motion in the physical plane. Choose this point as the origin of a fixed, Cartesian X-Y frame. The orbits of Γ_3 are defined for all positive values of the radius. We use the relation $\phi(\tau)$ defined above which we note is independent of the initial conditions of the orbit. Following the arguments in Section 2.3, we can construct a relation $\tau(X, Y)$ and hence a 1-form γ on the plane. The map as defined by (16) takes the form: $\tau = [2\pi/(\alpha_1 + \alpha_2 + \alpha_3)] \tan^{-1}(-Y/X)$ and the other quantities in the general formula (18) take on the following form:

$$\begin{aligned}
\Omega &= \frac{\alpha_1 + \alpha_2}{2\pi}, \\
f &= \frac{\alpha_3}{2\pi} \frac{-\epsilon r^2 \sin(\tilde{\phi} - \theta) + r \sin\left[2(\tilde{\phi} - \theta)\right]}{1 - 2\epsilon r \cos(\tilde{\phi} - \theta) + \epsilon^2 r^2}, \\
f_0 &= \frac{\alpha_3}{2\pi} \sin\left[2(\tilde{\phi} - \theta_0)\right] = \frac{\alpha_3}{2\pi} \sin\left[2\left(\tilde{\phi} - \frac{\alpha_1 t + \alpha_2 t}{2\pi} - \tilde{\theta}_0\right)\right], \\
\int f_0\, dt_{|t=0} &= \frac{\alpha_3}{2(\alpha_1 + \alpha_2)} \cos 2\theta_i.
\end{aligned}$$

Hence,

$$\gamma = \left(\frac{Y}{X^2 + Y^2} dX - \frac{X}{X^2 + Y^2} dY\right) \frac{\Gamma_3}{\Gamma_1 + \Gamma_2 + \Gamma_3} \cos 2\theta_i \tag{28}$$

is the 1-form defined on the plane whose integral around the closed path of Γ_3 gives (26). The geometric phase for this problem is therefore given by

$$\theta_g = \oint \gamma,$$

where γ is given by (28).

Let the radius of the circular orbit of Γ_3 be denoted by R. Since $X^2 + Y^2 = R^2$ and $\int (YdX - XdY) = 2A_v$ (using Stokes' Theorem) where $A_v = \pi R^2$ is the area swept out by the orbit of Γ_3, we get another expression for the geometric phase in this problem,

$$\begin{aligned} \theta_g = \oint \gamma \ &= \ \frac{2A_v}{R^2} \frac{\Gamma_3}{\Gamma_1 + \Gamma_2 + \Gamma_3} \cos 2\theta_i, \\ &= \ \frac{8\pi^2 A_v}{L^2} \frac{\Gamma_3}{\Gamma_1 + \Gamma_2 + \Gamma_3} \cos 2\theta_i, \end{aligned}$$

where L is the circumference of the circle swept out by Γ_3, i.e., $L = 2\pi R$.

4. A Point Vortex in a Circle

4.1. Asymptotic Derivation

In this problem we consider the influence of a rigid boundary in a simple problem. We track a fluid particle orbiting a point vortex in a circle as in Figure 3. The vortex is stationary when at the center of the circle and moves in a concentric circular orbit with constant speed when displaced from the center. See [59], [60] for some interesting recent work on this flow field.

The circular motion of the vortex is governed by the Hamiltonian $H_v(p, q) = \frac{\Gamma}{4\pi} \log[R_2^2 - (p^2 + q^2)]$, where p, q are the (canonical) coordinates of the vortex of strength Γ in a Cartesian frame centred at the origin of the circle and R_2 is the radius of the circle. If $R_1 (= \sqrt{p^2 + q^2})$ denotes the radius of the vortex motion, then its speed along its orbit is given by $\frac{\Gamma}{2\pi} \frac{R_1}{R_2^2 - R_1^2}$. From this it follows that the time period of the vortex motion is given by $T = \frac{4\pi^2}{\Gamma}(R_2^2 - R_1^2)$. The motion of the fluid particle is also governed by a Hamiltonian system of equations. For $R_1 = 0$, the vortex remains stationary at the center of the circle and the particle motion is not affected by the boundary since the boundary is a streamline for the time-independent flow. It orbits the vortex in circular motion just as in an unbounded flow. If $R_1 \neq 0$, for (canonical) coordinates x, y in the same frame as above, the Hamiltonian is

$$H(x, y, t) = \frac{\Gamma}{4\pi} \log\left[\frac{(x-p)^2 + (y-q)^2}{(x-ap)^2 + (y-aq)^2}\right], \qquad a = \left(\frac{R_2}{R_1}\right)^2,$$

where the explicit time dependency of the Hamiltonian is due to the motion of the vortex. It is clear from the form of the Hamiltonian that the equivalent flow without boundary can be achieved by placing an image vortex (the farfield vortex) of strength $-\Gamma$ at radius R_2^2/R_1. The distance between the given vortex and the image vortex is, therefore, $(a-1)R_1 = D$ at all times. The time period of the vortex motion can be rewritten in terms of this distance as $T = \frac{4\pi^2}{\Gamma}(a-1)R_1^2$.

For the phase calculation, we convert to the relative coordinates $\hat{r}, \theta$ as in the previous problem:

$$\begin{aligned} x - p \ &= \ -\hat{r}\cos\theta, \qquad x - ap = D\cos\phi - \hat{r}\cos\theta, \\ y - q \ &= \ \hat{r}\sin\theta, \qquad y - aq = -D\sin\phi + \hat{r}\sin\theta, \end{aligned}$$

where $\phi = \frac{\Gamma}{2\pi}\frac{\hat{t}}{(R_2^2-R_1^2)} = \frac{\Gamma}{2\pi}\frac{\hat{t}}{(a-1)R_1^2}$ is the angular velocity of the vortex. The small parameter for the adiabatic problem is chosen, as before, as being proportional to the ratio of the initial particle-vortex distance, R_i, to the distance between the parent vortex and farfield vortex D. Specifically,

$$\epsilon = \frac{R_i}{D}.$$

The scheme for nondimensionalising the variables is the same as before, i.e., $\hat{r}$ by R_i and $\hat{t}$ by ω where ω is proportional to the frequency of the particle about an isolated vortex in the absence of the boundary, i.e., to $\frac{1}{R_i^2}$. This defines the nondimensional constant α as before. The nondimensional time period of the vortex motion becomes

$$T = \frac{4\pi^2}{\alpha(a-1)\epsilon^2} = \frac{\beta}{\epsilon^2},$$

and the slowly varying angle coefficient $\phi = \frac{\alpha(a-1)}{2\pi}\epsilon^2 t$. The nondimensional equations of motion for the particle are

$$\frac{dr}{dt} = \frac{\alpha}{2\pi}\left[\epsilon\sin(\phi-\theta) - \frac{\epsilon\sin(\phi-\theta)}{1-2\epsilon r\cos(\phi-\theta)+\epsilon^2 r^2}\right],$$

$$\frac{d\theta}{dt} = \frac{\alpha}{2\pi}\left[\frac{1}{r^2} + \frac{\frac{\epsilon}{r}\cos(\phi-\theta)-\epsilon^2}{1-2\epsilon r\cos(\phi-\theta)+\epsilon^2 r^2} - \frac{\epsilon\cos(\phi-\theta)}{r}\right],$$

with initial conditons $r(0) = 1, \theta(0) = \theta_i$.

These equations have the general structure of (8) and (9) discussed in Section 2.2. We choose the slow timescale $\tau = \epsilon^2 t$, verify that the assumptions made in that section hold here, and make a direct comparison of terms. We have

$$f = \frac{\alpha}{2\pi}\frac{\epsilon r^2\sin(\tilde{\phi}-\theta) - r\sin\left[2(\tilde{\phi}-\theta)\right]}{1-2\epsilon r\cos(\tilde{\phi}-\theta)+\epsilon^2 r^2},$$

$$\Omega = \frac{\alpha}{2\pi}.$$

Hence,

$$f_0 = f\left(r=1, \theta = \Omega t + \tilde{\theta}_0, \tilde{\phi}, \epsilon = 0\right),$$

$$= -\frac{\alpha}{2\pi}\sin\left[2\left(\tilde{\phi} - \frac{\alpha t}{2\pi} - \tilde{\theta}_0\right)\right].$$

Hence,

$$\int f_0\, dt_{|t=0} = -\frac{\cos 2\theta_i}{2},$$

and the slow phase term from (12) is

$$\theta_S \equiv \tilde{\theta}_0(\tau) = -\frac{\alpha}{2\pi}\tau\cos 2\theta_i + \theta_i.$$

Therefore, the Hannay-Berry phase in this problem using (15) is

$$\begin{aligned}\theta_g &= \int_0^{\beta} \frac{d\theta_S}{d\tau}\, d\tau, \\ &= -\frac{2\pi}{(a-1)} \cos 2\theta_i\,, && (29) \\ &= -\frac{2\pi}{(\frac{R_2}{R_1})^2 - 1} \cos 2\theta_i\,. && (30)\end{aligned}$$

4.2. Geometric Interpretation

Here, the vortex orbits are circular with common center at the center of the physical domain, i.e., the circle. The orbits of radii R_1 fill the punctured (open) disk $0 < R_1 < R_2$. We choose the center of the circle as the origin of our fixed, Cartesian X-Y frame. The orbit-dependent relation is given by $\phi(\tau) = \frac{\alpha}{2\pi}(a-1)\tau$. Following Section 2.3 the relation $\tau(X, Y)$ as defined by (16) takes the form $\tau(X, Y) = \frac{2\pi}{\alpha(a-1)} \tan^{-1}(-Y/X)$, and the 1-form on the disk as defined by (18) is

$$\gamma = -\frac{\cos 2\theta_i}{(a-1)} \left(\frac{Y}{X^2+Y^2} dX - \frac{X}{X^2+Y^2} dY \right). \tag{31}$$

Its integral around the circular vortex orbit gives the Hannay-Berry phase (29). Hence, the geometric phase in this problem is given by

$$\theta_g = \oint \gamma,$$

where γ is given by (31). Another expression for the phase is obtained in a similiar way as in the three-vortex problem (using Stokes' Theorem),

$$\begin{aligned}\theta_g = \oint \gamma &= -\frac{2A_v}{(a-1){R_1}^2} \cos 2\theta_i\,, \\ &= -\frac{8A_v\pi^2}{(a-1)L^2} \cos 2\theta_i\,,\end{aligned}$$

where $A_v = \pi R_1^2$ is the area of the vortex circle, with circumference $L = 2\pi R_1$.

5. A Mixing Layer Model

5.1. The Model

Consider an infinite number of equally spaced point vortices of the same strength and sign. The vortices in this configuration are all stationary due to the symmetry of the flow field about horizontal and vertical axes passing through any point vortex. If the strength of each vortex is Γ and the spacing between vortices is a, then the complex potential of the flow is given by (see [58], p. 224)[16]

$$w(z) = -\frac{i\Gamma}{2\pi} \log \sin\left[\pi\left(\frac{z - z_{vor}}{a}\right)\right],$$

where $z = x + iy$ is an arbitrary point in the flow field and $z_{vor} = x_{vor} + iy_{vor}$ denotes the position of any *one* vortex in the row in some chosen x-y coordinate system. (The row is assumed to be parallel to the x-axis). The flow field is given by

$$\frac{dx}{dt} = \frac{\Gamma}{2a} \frac{\sinh\left[2\pi\left(\frac{y-y_{vor}}{a}\right)\right]}{\cosh\left[2\pi\left(\frac{y-y_{vor}}{a}\right)\right] - \cos\left[2\pi\left(\frac{x-x_{vor}}{a}\right)\right]}, \tag{32}$$

$$\frac{dy}{dt} = -\frac{\Gamma}{2a} \frac{\sin\left[2\pi\left(\frac{x-x_{vor}}{a}\right)\right]}{\cosh\left[2\pi\left(\frac{y-y_{vor}}{a}\right)\right] - \cos\left[2\pi\left(\frac{x-x_{vor}}{a}\right)\right]}. \tag{33}$$

Note that the equations of motion are invariant with respect to any transformation $z_{vor} \to z_{vor} \pm na$, $n = 1, 2, \ldots$. If the symmetry about the vertical axis is broken by a perturbation that pushes every adjacent pair of vortices towards each other by an amount $\triangle$, as shown in Figure 4, then the vortices start moving in pairs. They move such that each vortex in a pair orbits the other in a closed path. This motion is the same for every pair. The flow field can be viewed as the superposition of two infinite, evenly spaced rows [57], each with intervortex spacing $2a$. The motion of a vortex in row 1 is influenced only by the vortices in row 2 and vice versa.

We choose an x-y coordinate system in which the x-axis coincides with the direction of the original unperturbed row. Then if (x_{vor}, y_{vor}) are the coordinates of any one vortex in a row, the coordinates of any vortex in the opposite row are of the form

$$(2na - x_{vor}, -y_{vor}), \qquad n = 0, \pm 1, \pm 2, \ldots. \tag{34}$$

Since a point vortex moves like a fluid particle in its place, the equations of motion of any vortex in either row are given by (32) and (33) with a replaced by $2a$. Thus, the equations of motion for the perturbed vortices are

$$\frac{dx_{vor}}{dt} = \frac{\Gamma}{4a} \frac{\sinh\left[\frac{2\pi y_{vor}}{a}\right]}{\cosh\left[\frac{2\pi y_{vor}}{a}\right] - \cos\left[\frac{2\pi x_{vor}}{a}\right]}, \tag{35}$$

$$\frac{dy_{vor}}{dt} = -\frac{\Gamma}{4a} \frac{\sin\left[\frac{2\pi x_{vor}}{a}\right]}{\cosh\left[\frac{2\pi y_{vor}}{a}\right] - \cos\left[\frac{2\pi x_{vor}}{a}\right]}. \tag{36}$$

It is easily checked that the above system (35), (36) is Hamiltonian with

$$H = \frac{\Gamma}{8\pi} \log\left[\cosh\left(\frac{2\pi y_{vor}}{a}\right) - \cos\left(\frac{2\pi x_{vor}}{a}\right)\right],$$

---

[16] The complex potential is obtained by adding the complex potential due to each vortex, ignoring constant terms and using the relation $\sin z = \prod_{k=1}^{\infty}(1 - \frac{z^2}{k^2\pi^2})$ to simplify.

which is an invariant of the motion. Using this fact, an exact solution can be obtained for initial conditions $x_{vor}(0) = a/2 - \triangle$, $y_{vor}(0) = 0$ in terms of the Jacobi elliptic functions,

$$\tan\left(\frac{\pi x_{vor}}{a}\right) = \cot\left(\frac{\pi\triangle}{a}\right)\operatorname{cn}\left(\frac{\pi\Gamma t}{4ka^2}, k\right), \tag{37}$$

$$\tanh\left(\frac{\pi y_{vor}}{a}\right) = -\frac{\cot\left(\frac{\pi\triangle}{a}\right)}{1+\cot^2\left(\frac{\pi\triangle}{a}\right)}\operatorname{sd}\left(\frac{\pi\Gamma t}{4ka^2}, k\right), \tag{38}$$

where $k = \cos^2(\frac{\pi\triangle}{a})$ is the modulus of the associated (incomplete) elliptic integrals of the first kind [37]. Note that when $\triangle$ is small, k is close to 1 and when $\triangle$ is near $a/2$, i.e., when the vortices in each pair are close to each other, k is close to 0. The time period of the vortex motion, T, is related in a simple manner to the complete elliptic integral, K, as

$$T = \frac{16ka^2}{\pi\Gamma}K.$$

Since K has infinite series representations in terms of the modulus k [37], the time period T can also be represented exactly by an infinite series. We choose the following representation for K:

$$K = \frac{\pi}{2}\sum_{m=0}^{\infty}\frac{(\frac{1}{2})_m(\frac{1}{2})_m}{m!m!}k^{2m}, \tag{39}$$

where $(\frac{1}{2})_0 = 1$, $(\frac{1}{2})_m = \frac{1}{2}(\frac{1}{2}+1)\cdots(\frac{1}{2}+m-1)$ for $m = 1, 2, \ldots$, and $m!$ represents factorial of m. Hence,

$$\begin{aligned} T &= \frac{16ka^2}{\pi\Gamma}\left[\frac{\pi}{2}\sum_{m=0}^{\infty}\frac{(\frac{1}{2})_m(\frac{1}{2})_m}{m!m!}k^{2m}\right], \\ &= \frac{8a^2}{\Gamma}\left[\sum_{m=0}^{\infty}\frac{(\frac{1}{2})_m(\frac{1}{2})_m}{m!m!}k^{2m+1}\right]. \end{aligned}$$

The equations of motion, in x-y coordinates, for the particle in the perturbed field are shown below. These equations are obtained from (32) and (33) using relation (34),

$$\begin{aligned} \frac{dx}{dt} &= \frac{\Gamma}{4a}\frac{\sinh\left[\pi\left(\frac{y-y_{vor}}{a}\right)\right]}{\cosh\left[\pi\left(\frac{y-y_{vor}}{a}\right)\right] - \cos\left[\pi\left(\frac{x-x_{vor}}{a}\right)\right]} \\ &\quad + \frac{\Gamma}{4a}\frac{\sinh\left[\pi\left(\frac{y+y_{vor}}{a}\right)\right]}{\cosh\left[\pi\left(\frac{y+y_{vor}}{a}\right)\right] - \cos\left[\pi\left(\frac{x+x_{vor}}{a}\right)\right]}, \end{aligned} \tag{40}$$

$$\begin{aligned} \frac{dy}{dt} &= -\frac{\Gamma}{4a}\frac{\sin\left[\pi\left(\frac{x-x_{vor}}{a}\right)\right]}{\cosh\left[\pi\left(\frac{y-y_{vor}}{a}\right)\right] - \cos\left[\pi\left(\frac{x-x_{vor}}{a}\right)\right]} \\ &\quad - \frac{\Gamma}{4a}\frac{\sin\left[\pi\left(\frac{x+x_{vor}}{a}\right)\right]}{\cosh\left[\pi\left(\frac{y+y_{vor}}{a}\right)\right] - \cos\left[\pi\left(\frac{x+x_{vor}}{a}\right)\right]}, \end{aligned} \tag{41}$$

where (x_{vor}, y_{vor}) refer to the coordinates of any one vortex in either row.

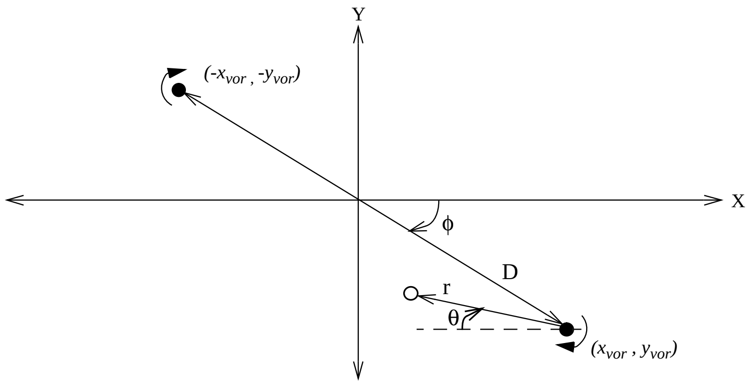

Fig. 5. An orbiting pair of vortices (filled circles) and a fluid particle (unfilled circle) in the shear layer model. The Cartesian $X - Y$ frame is centered at the midpoint of the line joining the vortices.

5.2. The Geometric Phase in the Model

For the geometric phase calculation in this model we focus on a pair of nearest vortices from opposite rows as shown in Figure 5. We use, as before, $\hat{r}$ for the particle-parent vortex distance[17] and $\hat{D}$ for the parent vortex-farfield vortex distance, and θ and ϕ for their respective angles with the horizontal axis. The origin of the x-y coordinate system (chosen earlier so that the x-axis coincided with the original unperturbed row) is now fixed midway between the two vortices. We thus obtain the transformations

$$
\begin{aligned}
x - x_{vor} &= -\hat{r}\cos\theta, \qquad 2x_{vor} = \hat{D}\cos\phi, \\
y - y_{vor} &= \hat{r}\sin\theta, \qquad 2y_{vor} = -\hat{D}\sin\phi.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
x + x_{vor} &= \hat{D}\cos\phi - \hat{r}\cos\theta, \\
y + y_{vor} &= -\hat{D}\sin\phi + \hat{r}\sin\theta.
\end{aligned}
$$

Denoting initial distances by r_i and $\hat{D}_i$, respectively (note that $D_i = a - 2\triangle$), we introduce nondimensional variables as before,

$$
\begin{aligned}
r &= \frac{\hat{r}}{r_i}, \\
D &= \frac{\hat{D}}{D_i}.
\end{aligned}
$$

For notational convenience, we introduce the nondimensional parameter $\delta = \pi \frac{D_i}{a}$. Thus,

[17] The parent vortex could be chosen to be either vortex of the pair.

$k = \sin^2(\delta/2)$. The small parameter ϵ is defined as

$$\epsilon = \delta \frac{r_i}{D_i}.$$

The time is nondimensionalised as

$$t = \omega \hat{t},$$

where ω, for fixed δ, is a frequency proportional to that of a particle around an isolated vortex in unbounded flow. We define it as

$$\begin{aligned} \omega &= \frac{\Gamma}{4\pi k r_i^2}, \\ &= \frac{\pi\Gamma}{4ka^2}\frac{1}{\epsilon^2}. \end{aligned} \tag{42}$$

The arguments of the elliptic functions in (37) and (38), thus transform as

$$\frac{\pi\Gamma\hat{t}}{4ka^2} = \epsilon^2\omega\hat{t} = \epsilon^2 t,$$

which is the slow timescale of the motion of the vortices. The nondimensional time period is

$$T = \frac{4K}{\epsilon^2}.$$

Note that this nondimensional time period depends on δ through K.

With these transformations the nondimensional equations of motion for the particle in (r, θ) become

$$\frac{dr}{dt} = k\epsilon \left[\frac{\begin{array}{c} S_h \cosh(\epsilon r \sin\theta)\cos\theta - C_h \sinh(\epsilon r \sin\theta)\cos\theta \\ -S_t \cos(\epsilon r \cos\theta)\sin\theta + C_t \sin(\epsilon r \cos\theta)\sin\theta \end{array}}{\begin{array}{c} -S_h \sinh(\epsilon r \sin\theta) + C_h \cosh(\epsilon r \sin\theta) \\ -S_t \sin(\epsilon r \cos\theta) - C_t \cos(\epsilon r \cos\theta) \end{array}} \right.$$

$$\left. + \frac{\sin(\epsilon r \cos\theta)\sin\theta - \sinh(\epsilon r \sin\theta)\cos\theta}{\cosh(\epsilon r \sin\theta) - \cos(\epsilon r \cos\theta)} + \frac{S_t \sin\theta - S_h \cos\theta}{C_h - C_t} \right], \tag{43}$$

$$\frac{d\theta}{dt} = \frac{k\epsilon}{r} \left[\frac{\begin{array}{c} -S_h \cosh(\epsilon r \sin\theta)\sin\theta + C_h \sinh(\epsilon r \sin\theta)\sin\theta \\ -S_t \cos(\epsilon r \cos\theta)\cos\theta + C_t \sin(\epsilon r \cos\theta)\cos\theta \end{array}}{\begin{array}{c} -S_h \sinh(\epsilon r sin\theta) + C_h \cosh(\epsilon r \sin\theta) \\ -S_t \sin(\epsilon r \cos\theta) - C_t \cos(\epsilon r \cos\theta) \end{array}} \right.$$

$$\left. + \frac{\sin(\epsilon r \cos\theta)\cos\theta + \sinh(\epsilon r \sin\theta)\sin\theta}{\cosh(\epsilon r \sin\theta) - \cos(\epsilon r \cos\theta)} + \frac{S_t \cos\theta + S_h \sin\theta}{C_h - C_t} \right], \tag{44}$$

where

$$\begin{aligned} S_h(\epsilon^2 t) &= \sinh(\delta D(\epsilon^2 t)\sin\phi(\epsilon^2 t)),\\ C_h(\epsilon^2 t) &= \cosh(\delta D(\epsilon^2 t)\sin\phi(\epsilon^2 t)),\\ S_t(\epsilon^2 t) &= \sin(\delta D(\epsilon^2 t)\cos\phi(\epsilon^2 t)),\\ C_t(\epsilon^2 t) &= \cos(\delta D(\epsilon^2 t)\cos\phi(\epsilon^2 t)), \end{aligned}$$

are functions of the vortex motion and $C_h - C_t = L$ is an invariant. For the chosen initial conditions of the vortex ($D(0) = 1, \phi(0) = 0$), $L = 1 - \cos\delta = 2k$. The initial conditions for the particle are $r(0) = 1, \theta(0) = \theta_i$.

The equations of motion as written above can be brought in the form of (8) and (9) by expanding (for small ϵ) the trigonometric and hyperbolic functions that appear in them. This enables us to rewrite them as

$$\begin{aligned} \frac{dr}{dt} &= \epsilon^2 f(r,\theta,D,\phi,\epsilon),\\ \frac{d\theta}{dt} &= \frac{2k}{r^2} + \epsilon^2 g(r,\theta,D,\phi,\epsilon), \end{aligned}$$

where f and g are infinite series obtained from the expansions. The series for f is of the following form:

$$\begin{aligned} f(r,\theta,D,\phi,\epsilon) &= kr\left[\frac{S_h S_t}{L^2}\cos 2\theta + \left(\frac{S_h^2 - S_t^2}{2L^2} - \frac{2}{3}\right)\sin 2\theta\right]\\ &\quad + \epsilon f_1(r,\theta,D,\phi) + \cdots. \end{aligned}$$

Denoting the slow time by $\tau = \epsilon^2 t$, we apply the multiscale method as outlined in Section 2.2. As per our notation we denote functions of τ alone by overhead $\sim$. We verify that the assumptions made in that section hold here and by a direct comparision of terms we get

$$\begin{aligned} \Omega &= 2k,\\ f_0 &= f(r=1, \theta = \Omega t + \tilde{\theta}_0, \tilde{D}, \tilde{\phi}, \epsilon = 0),\\ &= k\left[\frac{\tilde{S}_h\tilde{S}_t}{L^2}\cos\left[2\left(2kt + \tilde{\theta}_0\right)\right] + \left(\frac{\tilde{S}_h^2 - \tilde{S}_t^2}{2L^2} - \frac{2}{3}\right)\sin\left[2\left(2kt + \tilde{\theta}_0\right)\right]\right]. \end{aligned}$$

Hence,

$$\begin{aligned} \int f_0 dt_{|t=0} &= \frac{1}{4}\left[\frac{\tilde{S}_h(0)\tilde{S}_t(0)}{L^2}\sin 2\theta_i - \left(\frac{\tilde{S}_h^2(0) - \tilde{S}_t^2(0)}{2L^2} - \frac{2}{3}\right)\cos 2\theta_i\right],\\ &= \left(\frac{k+3}{24k}\right)\cos 2\theta_i. \end{aligned}$$

The slow phase term from (12) is

$$\theta_S \equiv \tilde{\theta}_0(\tau) = \left(\frac{k+3}{6}\right)\tau\cos 2\theta_i + \theta_i.$$

Hence, the Hannay-Berry phase in this problem as given by (15) is

$$\theta_g = \int_0^{4K} \frac{d\theta_S}{d\tau}\, d\tau, \\ = \left(\frac{2k+6}{3}\right) K \cos 2\theta_i ,$$

where K is given by (39). Note that K depends on δ. As $\delta \to 0$, i.e., as $\triangle \to a/2$, we get $k \to 0$ and $K \to \pi/2$ and the above phase approaches the value obtained by Newton [53] as a special case of the three-vortex problem, i.e., $\pi \cos 2\theta_i$.

5.3. Geometric Interpretation

The vortex orbits in this problem are obtained by varying the subharmonic perturbation $\triangle$ in the open interval $0 < \triangle < a/2$. These orbits fill a deleted neighbourhood U of their common center which lies midway between the vortices. The fixed, Cartesian X-Y frame is the same as in the previous section with origin at this point. As discussed in Section 2.3, we would like an explicit relation $\tau(X, Y)$ and hence a 1-form γ defined on U. However, unlike in the previous problems we cannot obtain explicit expressions. This is due to our inability to invert the orbit-dependent relation $\phi(\tau)$ and hence to get an explicit expression for G' as defined in Section 2.3. In principle, this relation can be obtained from the exact solutions of the vortex motion (37) and (38). Specifically, $\phi(\tau) = \tan^{-1}(-y_{vor}/x_{vor})$ where x_{vor} and y_{vor} are the inverse tangent and inverse hyperbolic tangent functions (respectively) defined by the exact solutions. It is clear from this that inverting $\phi(\tau)$ and obtaining an expression in terms of known analytic functions is difficult. We emphasize, however, that in principle a 1-form γ is still defined on U as per (17) whose integral around the closed vortex orbit gives the Hannay-Berry phase. In fact, in the limiting case in which $\triangle \to a/2$, the vortices pair in (approximately) circular orbits. The 1-form in this case is given explicitly by

$$\gamma = \left(\frac{Y}{X^2+Y^2} dX - \frac{X}{X^2+Y^2} dY\right) \frac{\cos 2\theta_i}{2},$$

and the geometric phase is

$$\theta_g = \oint \gamma = \pi \cos 2\theta_i ,$$

which is the formula obtained by Newton [53].

6. Conclusion

Our goal in this paper has been to analyse three canonical point vortex configurations of increasing complexity that exhibit geometric phases of the type now well documented in other mechanical contexts. In all three cases, two timescales enter into the problem because of our assumption that the phase object is close to the parent vortex compared to the farfield vortex. When nondimensionalised, the farfield vortices provide a slowly varying background field on the phase object and in the adiabatic limit $\epsilon \to 0$, simple formulas ((5),(6),(7)) for the geometric phase induced on the phase object are derived. A secondary emphasis has been on giving an asymptotic interpretation of these phases as arising from a balance between two competing terms in the limit as the small parameter $\epsilon \to 0$. Although this concept is implicit in other (adiabatic) geometric phase discussions, we have not seen it discussed explicitly. It follows naturally from this interpretation that a multiscale asymptotic method can be used for calculating the phase. In a sequel to this paper (part II), we will describe some of the physical consequences of these geometric phases.

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