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Vortex Motion and the Geometric Phase. Part II: Slowly Varying Spiral Structures

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Summary. We derive formulas for the long time evolution of passive interfaces in three "canonical" incompressible, inviscid, two-dimensional flow models. The point vortex models, introduced in Part 1 [1] are (i) a "restricted" three-vortex problem, (ii) a vortex and a particle in a closed circular domain, and (iii) a particle in the flowfield of a mixing layer model undergoing a vortex pairing instability. In each configuration, it was shown in Part 1 that the passive particle exhibits a geometric or Hannay-Berry phase over long time periods induced by the slowly varying periodic background field. In this paper we show how the formula for the evolution of a passive interface driven by the dynamics of the vortices inherits this geometric phase effect. The interface wraps into a spiral formation around the "parent" vortex, with a slowly varying component induced by the farfield vorticity. The length formula for the long time growth of the slowly rotating spiral decomposes into a "dynamic" part and a "geometric" part. The dynamic part is the length in the "unperturbed" system—i.e., in the absence of the background field—and the geometric part is the contribution of the geometric phase θ_g for a passive particle in the flow. We derive the following simple formula for an interface along a smooth curve joining two arbitrary particles labelled A and B. Define $\triangle L(T)$ as the difference in interface lengths between the "unperturbed" system and the "perturbed" slowly varying system at the end of the long time period T. In each case, $\triangle L = \lim_{T\to\infty} \triangle L(T) = -\int_{\xi_A}^{\xi_B} d(\xi\theta_g)$, where ξ is the radial coordinate parametrizing the interface at $t = 0$.

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1. Introduction

A frequently recurring theme in fluid dynamics literature is the focus on “spiral structures” as important dynamical states in both two-dimensional and three-dimensional flows. These structures are especially prominent in vortex-dominated flows [2] where they appear in the evolution of vorticity regions or interfaces of passive scalars (e.g., concentration, temperature). They form for several reasons, depending on the flow configuration in question. Coherent vortices can acquire spiral-like structures near their vorticity maximum due to the winding up of variations in the initial vorticity distribution. As the flow evolves, further spirals are created through instabilities or collisions. This idea has been used more than once as the basis for phenomenological models for turbulent flows—see, for example, Lundgren [3], Gilbert [5], Moffat [6].

In an influential paper by Lundgren [3], a three-dimensional spiral vortex model is introduced in which vorticity is not axially symmetric as in the Burger’s vortex (see Marcu, Meiburg, and Newton [4]), but has a characteristic spiral structure. In this model, these structures arise dynamically from the interaction of two regions of constant vorticity. As the vortices coalesce into a single vortex core, spiral arms are thrown out in an effort to conserve energy and angular momentum. Because of the differential rotation caused by the dominant vortex core, the spiral arms stretch and deform into thin vortex sheets which then dissipate out due to viscous diffusion. In Gilbert [5], the focus is on the winding up of a weak vortex patch by a strong vortex core, in two dimensions, where the patch is so weak that it can be treated as being passively advected. The winding is due to the differential rotation rates inherent to the vortex structures. Essentially, the model considered by Gilbert [5] is kinematic as opposed to dynamic.

In a different context, the nonlinear evolution of an incompressible shear layer has been studied extensively. In Corcos and Sherman [8] and in Pozrikidis and Higdon [9], one goal is to determine the growth rate of interfacial area between two separated fluid regions. This quantity is of great interest to chemical engineers who study reacting streams. If the reaction rate is fast, and hence diffusion is limited, the generation of products is proportional to the growth rate of the interfacial area. In Pozrikidis and Higdon [9], a vortex dynamics (inviscid) simulation of a 2-D shear layer is carried out. One of the conclusions of this work is that the growth rate of interfacial area approaches a constant value for all shear layers they consider, a fact consistent with the experiments of Breidenthal [10], who found constant reaction rates as long as the mixing layer maintains its two-dimensional structure. This paper should be read in conjunction with that of Corcos and Sherman [8], who perform a finite difference calculation on the corresponding viscous problem. In their relatively low Reynolds number simulation, there is a rapid diffusion of vorticity in contrast to the inviscid calculation of Pozrikidis and Higdon [9]. On the one hand, the inviscid calculations are capable of showing fine details (small-scales) of the flow that are obscured by viscous effects. On the other hand, the inviscid calculations are more difficult to compare with experiments since they represent an idealised limit. The work of Corcos and Sherman [8] goes further in identifying clearly the two distinct stages of the nonlinear evolution of a 2-D shear layer. The first stage is the roll-up of the interface around a local vorticity maximum. The interface evolves into a spiral formation where the marker particles migrate inward along the arms and accumulate near their center. The second stage is the by now well-documented

[11] vortex pairing process in which, due to a dominant subharmonic instability, neighbouring vortices pair and orbit each other as the spiral arms continue to wrap locally and evolve globally (see, for example, Figure 4 of Corcos and Sherman [8]). In general, the process is complex and well studied—see, for example, the review of Ho and Huerre [12].

It is not our intention in this paper to comment on the merits of the spiral vortex models as far as their relevance to turbulence theory is concerned. Rather, we focus more narrowly on a particular dynamical question associated with the evolution of spiral structures in flowfields populated with point vortices. We are interested in the long time growth of a passively advected interface in such flows under the influence of two processes. On the one hand, there is a "fast" wrapping of the interface into a spiral around a nearby vorticity maximum (point vortex). On the other hand, there is a slow evolution of the spiral interface due to the farfield vorticity. We thus have two widely separated timescales in the dynamics of the interface, and we consider the evolution of such a slowly varying spiral structure.

The flowfields that we consider were introduced in Part I [1], [13] in the context of geometric phases in Hamiltonian systems with slowly varying parameters (see references therein). We identified three simple point-vortex configurations that, in an adiabatic setting, exhibit a geometric phase in the evolution of a "phase object". The phase object, depending upon the configuration, is one of the point vortices of the configuration or a passive particle such as a fluid or tracer particle in the 2-D incompressible, inviscid flow. The phase holonomy is exhibited in the angle variable of the phase object as the configuration evolves under its natural dynamics at the end of one (appropriately defined) time period of the vortex motion. The adiabatic setting is defined by taking the phase object close to the parent vortex compared to its distance from the farfield vortices; a separation of timescales is then guaranteed.

In this paper, we examine a passively evolving, smooth interface in these flows in a similar adiabatic setting, i.e., in the vicinity of the parent vortex. We find that the effect of the slowly moving farfield vortices on the dynamics of the interface, although weak, accumulates over long times, and at the end of one farfield time period T gives an $O(1)$ contribution. In particular, we show that a simple formula emerges for the long time growth of the interface length. The formula shows that the growth decomposes into two distinct parts. The first, due to the rapid local wrapping about the parent vortex, is the growth in the absence of the farfield vortices, while the second is the $O(1)$ contribution of the slowly moving farfield vortices. The main result of this paper is to show that this $O(1)$ term can be written in terms of the appropriate geometric phase for a passive particle in the flow.

In the next section, we define the interface problem and describe the three vortex configurations in which we analyze it. Section 3 contains a description of our asymptotic procedure for the general equations of motion. The calculation is divided into three main steps whose details are outlined. In Section 4 we implement the procedure on the first two of our vortex configurations: (a) a restricted three-vortex setup, and (b) a vortex in a circular domain. These two configurations are somewhat special in that there is a uniformly rotating frame in which the flowfield is steady. In Section 5 we perform the calculations on a model for a mixing layer undergoing a vortex-pairing process—this flowfield is not steady in any uniformly rotating frame.

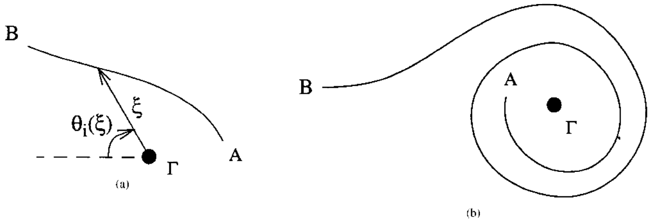

Fig. 1. (a) A passive interface between two particles labelled A and B in the flowfield of an isolated point vortex (filled circle) at time $t = 0$. (b) With time, the interface stretches and wraps around the vortex.

2. The Interface Problem

We view the passive interface as a smooth C^1 curve drawn in the flow domain, each point of which represents a passive particle at that location. Consider such an interface in the flowfield of an isolated point vortex of strength Γ at time $t = 0$, as shown in Figure 1a. Let the interface connect two arbitrary particles, labelled A and B. We choose a coordinate frame centered at the vortex location and parametrize the interface by ξ, the distance from the vortex, as shown in the figure. Assume that the interface is transversal at every point to the circular streamlines of the point vortex flow.[1] Consider now the evolution of an arbitrary particle $(r_0(t), \theta_0(t))$ on the interface with $r_0(0) = \xi$, $\xi_A \le \xi \le \xi_B$; $\theta(0) = \theta_i(\xi)$. We know that the time evolution of such a particle is governed by

$$\begin{aligned} r_0(t) &= \xi, \\ \theta_0(t) &= \frac{\Gamma t}{2\pi\xi^2} + \theta_i(\xi). \end{aligned}$$

This implies that with time the interface stretches and wraps around the vortex in a spiral as shown in Figure 1b. The arclength $L_0(t)$ of the interface is given by[2]

$$\begin{aligned} L_0(t) &= \int_{\xi_A}^{\xi_B} \sqrt{\left(r_0(t)\frac{d\theta_0(t)}{d\xi}\right)^2 + \left(\frac{dr_0(t)}{d\xi}\right)^2}\, d\xi \qquad (1) \\ &= \int_{\xi_A}^{\xi_B} \sqrt{1 + \left(\xi\frac{d\theta_0}{d\xi}\right)^2}\, d\xi \\ &= \int_{\xi_A}^{\xi_B} \sqrt{1 + \left(\xi\frac{d\theta_i}{d\xi} - \frac{\Gamma t}{\pi\xi^2}\right)^2}\; d\xi. \end{aligned}$$

Note that because of our transversality assumption, $d\theta_i/d\xi$ is finite at all points of the curve. It is then straightforward to expand the above expression for long times ($t \gg 1$)

[1] We show in the next section that relaxing this assumption gives a similar formula but with a different parametrization.

[2] We assume positive square roots everywhere in this paper.

to get the formula,

$$
\begin{aligned}
L_0(t) &= \int_{\xi_A}^{\xi_B} \sqrt{\left(\frac{\Gamma t}{\pi\xi^2}\right)^2 \left[1 - 2\frac{\xi\, d\theta_i/d\xi}{\Gamma t/\pi\xi^2} + \frac{1 + (\xi\, d\theta_i/d\xi)^2}{\left(\Gamma t/\pi\xi^2\right)^2}\right]}\; d\xi \\
&= \int_{\xi_A}^{\xi_B} \left[\frac{|\Gamma|\, t}{\pi\xi^2} - \operatorname{sgn}(\Gamma)\xi\frac{d\theta_i}{d\xi} + O\left(\frac{1}{t}\right)\right] d\xi \qquad (2) \\
&= \frac{|\Gamma|\, t}{\pi}\left(\frac{1}{\xi_A} - \frac{1}{\xi_B}\right) - \operatorname{sgn}(\Gamma)\int_{\xi_A}^{\xi_B} \xi\,\frac{d\theta_i}{d\xi} d\xi + O\left(\frac{1}{t}\right). \qquad (3)
\end{aligned}
$$

In the special case where the initial interface lies on a ray, $d\theta_i/d\xi = 0$, the above formula gives

$$
L_0(t) = \frac{|\Gamma|\, t}{\pi}\left(\frac{1}{\xi_A} - \frac{1}{\xi_B}\right) + O\left(\frac{1}{t}\right).
$$

It is interesting to note, as mentioned in the introduction, that the simulations of Pozrikidis and Higdon [9] and the experiments of Breidenthal [10] also predict linear growth of interfacial area for long times.

Now consider what happens to such an interface when subjected to an additional slowly varying background field; hence, allow the coordinates $(r(t,\tau), \theta(t,\tau))$ to depend on some slow timescale $\tau = \epsilon^2 t$ $(0 < \epsilon \ll 1)$. In addition, suppose the background field is periodic with period $T \sim 1/\epsilon^2$. In such a field, the "perturbed" interface length is given by

$$
L_\epsilon(t) = \int_{\xi_A}^{\xi_B} \sqrt{\left(r\frac{d\theta}{d\xi}\right)^2 + \left(\frac{dr}{d\xi}\right)^2}\, d\xi. \qquad (4)
$$

At any given time, we can compute the difference between the "unperturbed" length $L_0(t)$ and the "perturbed" length $L_\epsilon(t)$:

$$
\triangle L(t) = L_\epsilon(t) - L_0(t).
$$

From this, we can derive a formula for $\triangle L(t = T \sim 1/\epsilon^2)$:

$$
\triangle L(T) = L_\epsilon(T) - L_0(T). \qquad (5)
$$

We now ask the question: What is $\triangle L(T)$ in the limit as $\epsilon \to 0$ or, equivalently, as

$T \to \infty$? It is not difficult to see that, in general, the interfacial growth in the presence of the background field is due to two distinct but highly coupled effects:

(i) a 'fast' wrapping of the interface in a spiral around the vortex,
(ii) a 'slow' evolution of the spiral due to the background field.

The interaction and balance of these two effects is, of course, what determines $\triangle L(T)$.

In this paper we answer the above question for the three canonical point-vortex configurations that we investigated in Part I. As mentioned in the previous section, in each of these configurations we tracked a phase object in the vicinity of a parent vortex. The slowly varying background field is provided by the farfield vortices. We showed that the phase object exhibits a geometric phase θ_g in its angle variable at the end of one time period T of the background field. In this paper, we take the phase object as a passive particle and show that in all three problems for a passive interface in the vicinity of the parent vortex:

$$\triangle L := \lim_{\substack{\epsilon \to 0 \\ (T \to \infty)}} \triangle L(T) = \lim_{\substack{\epsilon \to 0 \\ (T \to \infty)}} (L_\epsilon(T) - L_0(T)) = O(1). \tag{6}$$

This result is due to the fact that, in the three problems, as $\epsilon \to 0$ the background flow gets slower *and* weaker. This means that there is a balance between two compensating effects—the vanishing of the perturbation due to the background flow versus the increasing time period over which it acts. We will show that this balance can be directly related to the geometric or Hannay-Berry phase θ_g for a passive particle in these flows by the following simple formula:

$$\triangle L = -\int_{\xi_A}^{\xi_B} d(\xi \theta_g).$$

This formula shows two things:

(i) $\triangle L$ depends only on geometric quantities, and in particular, on the geometric phase for the appropriate vortex configuration. It is independent of the frequency of revolution of the vortices.
(ii) $\triangle L$ is a path-independent quantity and hence depends only on the endpoints (A, B), not on the shape of the interface connecting A and B, as long as the interface is everywhere transverse to the circular streamlines of the isolated parent vortex.

We briefly describe the point-vortex configurations and summarize the results we obtain for $\triangle L$ below. In all formulas, θ_i refers to the initial angle of the phase object.

Configuration 1. A Restricted Three-Vortex Problem. In Part I we considered three-point vortices in an unbounded plane. Their strengths are of the same sign, but can be of arbitrary magnitude $(\Gamma_1, \Gamma_2, \Gamma_3)$. Without loss of generality, we took Γ_1 as the parent vortex, Γ_2 as the phase object, and Γ_3 as the farfield vortex. The geometric phase induced on the phase object was calculated in Part I (equation (5)). For the purpose of this paper we take $\Gamma_2 = 0$ and the geometric phase on the passive particle is then given by

$$\theta_g = \frac{\Gamma_3}{\Gamma_1 + \Gamma_3} 2\pi \cos 2\theta_i.$$

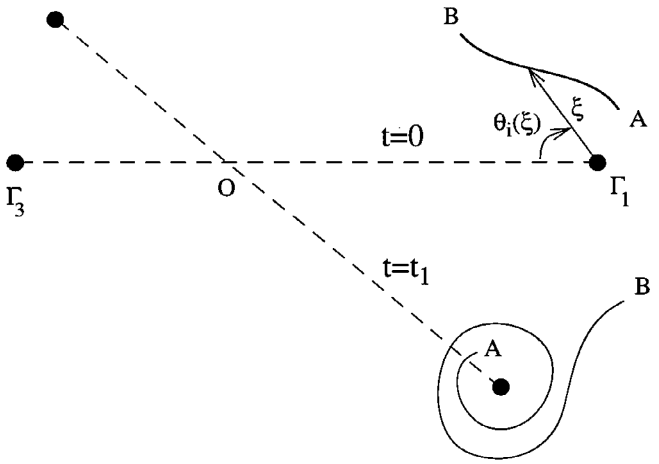

Fig. 2. Two like-signed point vortices (filled circles) and a passive interface between two particles labelled A and B close to Γ_1. As the vortices rotate uniformly about the center of vorticity O, the interface stretches and wraps around Γ_1.

The appropriate configuration with interface is shown in Figure 2. We will show that, for this flow,

$$\begin{aligned}\triangle L &= -\int_{\xi_A}^{\xi_B} d(\xi\theta_g)\\ &= \left(\frac{4\pi\Gamma_3}{\Gamma_1+\Gamma_3}\right)\int_{\xi_A}^{\xi_B}\left(\xi\sin 2\theta_i\frac{d\theta_i}{d\xi}-\frac{\cos 2\theta_i}{2}\right)d\xi.\end{aligned}$$

Configuration 2. A Vortex and a Particle in a Circle. In this configuration we consider a point vortex inside a circular domain. The vortex in any eccentric position moves in a closed circular path with radius R_1 and with constant frequency. We consider a passive particle orbiting this parent vortex Γ. The farfield vorticity is due to the circular boundary of radius $R_2 > R_1$. Equivalently, we can think of the farfield vortex as an image vortex $-\Gamma$ placed at its image point R_2^2/R_1 outside the circle. The flow configuration with interface is shown in Figure 3. The geometric phase for the particle was shown in Part I to be

$$\theta_g = -\frac{2\pi\cos 2\theta_i}{(R_2/R_1)^2-1}.$$

We will show that, for this flow,

$$\begin{aligned}\triangle L &= -\int_{\xi_A}^{\xi_B} d(\xi\theta_g)\\ &= \left(\frac{4\pi}{(R_2/R_1)^2-1}\right)\int_{\xi_A}^{\xi_B}\left(-\xi\sin 2\theta_i\frac{d\theta_i}{d\xi}+\frac{\cos 2\theta_i}{2}\right)d\xi.\end{aligned}$$

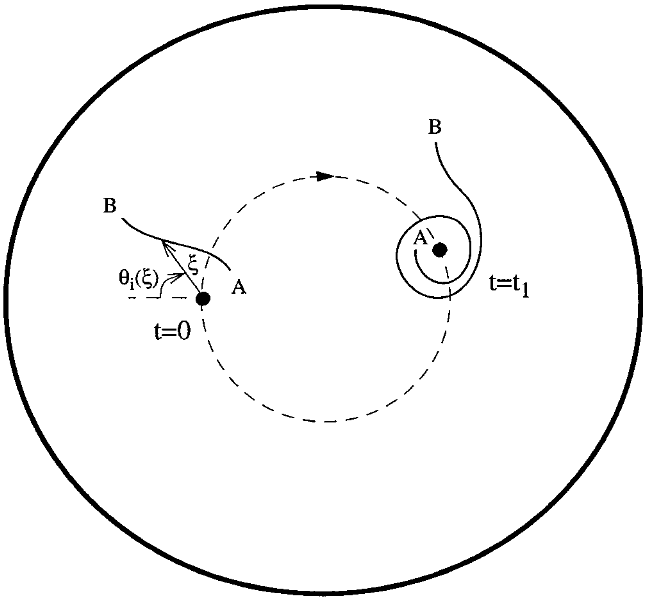

Fig. 3. A point vortex (filled circle) in a circular domain (solid circle), and a passive interface between two particles labelled A and B close to it. As the vortex moves in a circular orbit (dashed circle) about the center of the domain, the interface stretches and wraps around it.

Configuration 3. A Particle in a Mixing Layer Model. In this configuration, an infinite row of evenly spaced, equal strength vortices is given a subharmonic perturbation so that neighboring vortices pair up and undergo periodic motion. We consider a passive particle near any parent vortex. The farfield flow is due to the infinite number of other point vortices periodically spaced. For this flow, the geometric phase for the particle is given by

$$\theta_g = \left(\frac{2k+6}{3}\right) K \cos 2\theta_i ,$$

where k is the modulus of the elliptic function solutions of the periodic vortex motion and K is the complete elliptic integral associated with these solutions and related to the time period of the vortex motion. See Part I for more details. The flow configuration with interface is shown in Figure 4. In this paper, we will prove that

$$\begin{aligned} \triangle L &= -\int_{\xi_A}^{\xi_B} d(\xi\theta_g) \\ &= \int_{\xi_A}^{\xi_B} \left[-\left(\frac{2k+6}{3}\right) K \cos 2\theta_i + \left(\frac{4k+12}{3}\right) K \xi \sin 2\theta_i \frac{d\theta_i}{d\xi} \right] d\xi . \end{aligned}$$

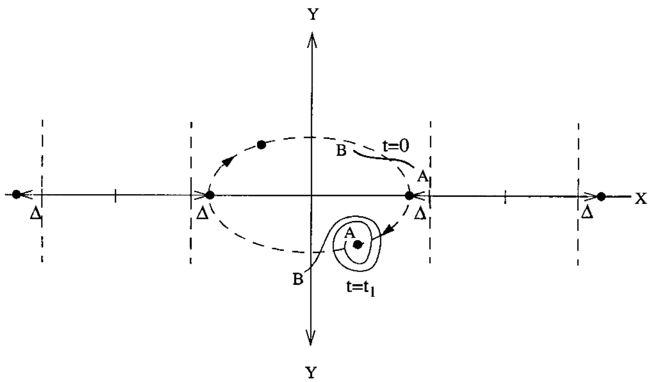

Fig. 4. A mixing layer model in which an infinite row of equality spaced point vortices (filled circles; vertical dashed lines mark the initial positions of the vortices) is given a subharmonic perturbation (of magnitude Δ and direction shown by arrows along X-axis). The subsequent motion of the of the vortices is shown in the central window. This motion is identical in every other window. As the vortices move in closed orbits, a passive interface between two particles labelled A and B near one of the vortices gets stretched and wrapped around the vortex.

3. Asymptotic Procedure

In this section, we outline the method to compute the interface formulae for the general equations introduced in Part I. We also highlight and summarize the main steps in the computation and mention the specific assumptions made on the behaviour of higher order terms in deriving our final result. The general form of the equations we consider are given by

$$\frac{dr}{dt} = \epsilon^2 f(r, \theta, D(\epsilon^2 t), \phi(\epsilon^2 t), \epsilon),$$

$$\frac{d\theta}{dt} = \frac{\Omega}{r^2} + \epsilon^2 g(r, \theta, D(\epsilon^2 t), \phi(\epsilon^2 t), \epsilon),$$

with initial conditions $r(0) = 1$ and $\theta(0) = \theta_i$.

Here, (r, θ) denote the nondimensional polar coordinates of the phase object with respect to the parent vortex, (D, ϕ) are nondimensional polar variables representing the periodic vortex motion, f, g are the components of the vector field due to the farfield vortices, and ϵ is the perturbation parameter. For small ϵ, we introduce a slow timescale $\tau = \epsilon^2 t$ and use a multiscale method to obtain asymptotic solutions. Subject to certain conditions on f and g (see Part I), we have nondimensional asymptotic solutions of the form

$$r(t, \tau; \theta_i) = 1 + \epsilon^2 r_2(t, \tau; \theta_i) + O(\epsilon^3), \tag{7}$$

$$\theta(t, \tau; \theta_i) = \Omega t + \left(2\Omega\tau \int f_0 dt_{|t=0} + \theta_i\right) + \epsilon\tilde{\theta}_1(\tau; \theta_i) + \epsilon^2 \theta_2(t, \tau; \theta_i) + O(\epsilon^3), \tag{8}$$

where f_0 is the leading term in the Taylor expansion of f about $\epsilon = 0$, the integration is with respect to the fast time t, and Ω is a nondimensional constant. We have emphasized the dependence of the variables on $\theta_i(\xi)$, which need not be constant along the interface. We assume, for the present, that $d\theta_i/d\xi < \infty$ at all points of the interface, i.e., the initial interface is transversal at all points to the circular streamlines of the parent vortex flow. We relax this transversality assumption at the end of the section and show that it does not alter our result.

To evaluate $\triangle L$ from the above solutions, we proceed in three steps, as follows:

1. Write the asymptotic expansions for $r(t, \tau; \theta_i)$, $\theta(t, \tau; \theta_i)$ in dimensional form.
2. Compute $L_\epsilon(T)$ and formulate the difference:

$$\triangle L(T) = L_\epsilon(T) - L_0(T).$$

3. Take the limit $\epsilon \to 0$ to derive the expression $\triangle L$.

Step 1. The nondimensional variables in (7) and (8) are

$$r = \frac{\hat{r}}{\xi}, \qquad t = \omega\hat{t}, \qquad \tau = \epsilon^2 t = \epsilon^2\omega\hat{t}. \tag{9}$$

This gives

$$\begin{aligned} \hat{r}(\hat{t}, \xi) &= \xi\left[1 + \epsilon^2 r_2(\omega\hat{t}, \epsilon^2\omega\hat{t}, \theta_i(\xi)) + O(\epsilon^3)\right], \\ \theta(\hat{t}, \xi) &= \Omega\omega\hat{t} + \left(2\Omega\omega\epsilon^2\hat{t}\int f_0\, dt_{|t=0} + \theta_i\right) + \epsilon\tilde{\theta}_1(\epsilon^2\omega\hat{t}, \theta_i(\xi)) \\ &\quad + \epsilon^2\theta_2(\omega\hat{t}, \epsilon^2\omega\hat{t}, \theta_i(\xi)) + O(\epsilon^3), \end{aligned}$$

where $\omega \propto 1/\xi^2$ and $\epsilon = a_1\xi/D$ (a_1 constant). The limit $\epsilon \to 0$ can obviously be interpreted in two different ways: (i) ξ bounded above and below, $D \to \infty$, or (ii) D fixed, $\xi \to 0$. For calculating $\triangle L$, however, these two limits are *not* equivalent. Indeed, the second limit process does not give a well-defined interface problem. We therefore perform our calculation using the first limit process.

Step 2. We rewrite the above series at the end of the time period T of the farfield vortices. Since $T = a_2 D^2$ (a_2 constant), this implies that all terms involving the slow time ($\epsilon^2\omega\hat{t}$) appear as constants. In particular, the second term in the θ series above gives the geometric phase θ_g. The first term in the series is the angle swept out in time T of a particle about an isolated parent vortex of strength Γ in an unbounded flow. Indeed, in all three problems $\Omega\omega = \Gamma/(2\pi\xi^2)$, which is the angular frequency of such a motion. Defining $c = a_1^2 a_2$, one then gets for large T,

$$\hat{r}(T, \xi) = \xi\left[1 + \frac{c\xi^2}{T} r_2\left(\frac{T}{\xi^2}, \theta_i(\xi)\right) + O\left(\frac{1}{T\sqrt{T}}\right)\right], \tag{10}$$

$$\theta(T,\xi) = \frac{\Gamma T}{2\pi\xi^2} + \theta_g(\theta_i(\xi)) + \theta_i(\xi) + \frac{\sqrt{c}\xi}{\sqrt{T}}\theta_1(\theta_i(\xi)) + \frac{c\xi^2}{T}\theta_2\left(\frac{T}{\xi^2}, \theta_i(\xi)\right) + O\left(\frac{1}{T\sqrt{T}}\right). \quad (11)$$

The functions r_j, θ_j are now viewed as functions of T/ξ^2 and ξ. We assume that all these functions (with the exception of θ_0) are bounded in T/ξ^2 and hence $O(1)$ in T. This assumption implies and follows from the assumption of boundedness in t of the formal expansions (7) and (8). Denoting T/ξ^2 by ρ, we differentiate[3] the above series with respect to ξ to get the representations

$$\frac{d\hat{r}}{d\xi}(T,\xi) = 1 - 2c\frac{\partial r_2}{\partial\rho} + O\left(\frac{1}{\sqrt{T}}\right), \quad (12)$$

$$\frac{d\theta}{d\xi}(T,\xi) = -\frac{\Gamma}{\pi\xi^3}T + \frac{d}{d\xi}\left(\theta_g + \theta_i\right) - \frac{2c}{\xi}\frac{\partial\theta_2}{\partial\rho} + O\left(\frac{1}{\sqrt{T}}\right). \quad (13)$$

We assume that these derivatives are also bounded in T/ξ^2 and hence $O(1)$ in T. We then obtain the following representations:

$$\left[\hat{r}\frac{d\theta}{d\xi}\right]^2 = \left[-\frac{\Gamma}{\pi\xi^2}T + \left\{-\frac{cr_2\Gamma}{\pi} + E\right\} + O\left(\frac{1}{\sqrt{T}}\right)\right]^2$$
$$= \left\{\frac{\Gamma}{\pi\xi^2}\right\}^2 T^2 + 2\left\{\frac{\Gamma}{\pi\xi^2}\right\}\left\{\frac{cr_2\Gamma}{\pi} - E\right\}T + O(\sqrt{T}),$$

$$\left[\frac{d\hat{r}}{d\xi}\right]^2 = O(1),$$

where $E = \xi d(\theta_g + \theta_i)/d\xi - 2c(\partial\theta_2/\partial\rho)$. Hence,

$$\sqrt{\left[\hat{r}\frac{d\theta}{d\xi}\right]^2 + \left[\frac{d\hat{r}}{d\xi}\right]^2} = \sqrt{\left\{\frac{\Gamma}{\pi\xi^2}\right\}^2 T^2\left[1 + \frac{2\left\{\frac{cr_2\Gamma}{\pi} - E\right\}}{\left\{\frac{\Gamma}{\pi\xi^2}\right\}}\frac{1}{T} + O\left(\frac{1}{T\sqrt{T}}\right)\right]}$$
$$= \left|\frac{\Gamma T}{\pi\xi^2}\right|\left[1 + \frac{\left\{\frac{cr_2\Gamma}{\pi} - E\right\}}{\left\{\frac{\Gamma}{\pi\xi^2}\right\}}\frac{1}{T} + O\left(\frac{1}{T\sqrt{T}}\right)\right]$$
$$= \frac{|\Gamma|\,T}{\pi\xi^2} + \left\{\frac{cr_2\,|\Gamma|}{\pi} - \operatorname{sgn}(\Gamma)E\right\} + O\left(\frac{1}{\sqrt{T}}\right). \quad (14)$$

The length integral (4) then assumes the form

$$L_\epsilon(T) = \int_{\xi_A}^{\xi_B}\left[\frac{|\Gamma|\,T}{\pi\xi^2} + \left\{\frac{cr_2\,|\Gamma|}{\pi} - \operatorname{sgn}(\Gamma)E\right\} + O\left(\frac{1}{\sqrt{T}}\right)\right]d\xi.$$

[3] Note that differentiating and integrating these functions with respect to ξ changes their order with respect to T.

Subtracting $L_0(T)$ (as given by (2)) from this and writing out E gives

$$\triangle L(T) = \int_{\xi_A}^{\xi_B} \left[\frac{cr_2\,|\Gamma|}{\pi} + \mathrm{sgn}(\Gamma) \left(-\xi \frac{d\theta_g}{d\xi} + 2c\frac{\partial \theta_2}{\partial \rho} \right) + O\left(\frac{1}{\sqrt{T}} \right) \right] d\xi.$$

The functions $r_2(t, \tau, \theta_i)$ and $\theta_2(t, \tau, \theta_i)$ in all three problems are of the form $r_2(t, \tau, \theta_i) = r_{2F}(t, \tau, \theta_i) + \tilde{r}_2(\theta_i)$ and $\theta_2(t, \tau, \theta_i) = \theta_{2F}(t, \tau, \theta_i) + \tilde{\theta}_2(\tau, \theta_i)$, where the subscript F denotes the dependency on the fast time. This means that $r_2(T/\xi^2, \xi) = r_{2F}(T/\xi^2, \xi) + \tilde{r}_2(\xi)$ and $\theta_2(T/\xi^2, \xi) = \theta_{2F}(T/\xi^2, \xi) + \tilde{\theta}_2(\xi)$. Hence,

$$\begin{aligned} \triangle L(T) \;=\; & \int_{\xi_A}^{\xi_B} \left[\frac{c\tilde{r}_2\,|\Gamma|}{\pi} - \mathrm{sgn}(\Gamma) \cdot \xi \frac{d\theta_g}{d\xi} + \frac{cr_{2F}\,|\Gamma|}{\pi} \right. \\ & \left. + \,\mathrm{sgn}(\Gamma) \cdot 2c \frac{\partial \theta_{2F}}{\partial \rho} + O\left(\frac{1}{\sqrt{T}} \right) \right] d\xi. \end{aligned}$$

The first term in the integrand is related to the geometric phase in the problem by the following linear relation derived in Part I (equation (14)):

$$\theta_g = -2\Omega\beta\tilde{r}_2,$$

where β is a constant related to the dimensional time period as $T = \beta/\omega\epsilon^2$. This implies that

$$\begin{aligned} \theta_g \;&=\; -2\Omega\omega\epsilon^2 T\tilde{r}_2 \\ &=\; -2\Omega\omega\xi^2 c\tilde{r}_2 \\ &=\; -\frac{\Gamma}{\pi\xi^2}\xi^2 c\tilde{r}_2 \\ &=\; -\frac{\Gamma}{\pi} c\tilde{r}_2. \end{aligned}$$

Therefore,

$$\begin{aligned} \triangle L(T) \;=\; & \int_{\xi_A}^{\xi_B} \left[-\,\mathrm{sgn}(\Gamma) \left(\theta_g + \xi \frac{d\theta_g}{d\xi} \right) + \frac{cr_{2F}\,|\Gamma|}{\pi} \right. \\ & \left. + \,\mathrm{sgn}(\Gamma) \cdot 2c \frac{\partial \theta_{2F}}{\partial \rho} + O\left(\frac{1}{\sqrt{T}} \right) \right] d\xi. \end{aligned}$$

Step 3. The first two terms in the integrand depend only on ξ and are clearly the contribution of the geometric phase. The third and fourth terms in all three problems are the trignometric functions Sine or Cosine with arguments of the form $CT/\xi^2 + u(\theta_i)$, where C is a constant and u is a function of $\theta_i(\xi)$ alone. A simple application of the Riemann-Lebesgue lemma then shows that

$$\lim_{T\to\infty} \int_{\xi_A}^{\xi_B} \exp\left[i \left(CT/\xi^2 + u(\theta_i) \right) \right] d\xi \;=\; 0.$$

Assuming that all higher order terms in the integrand series display this behaviour,[4] we get

$$\triangle L = \lim_{T\to\infty} \triangle L(T) = -\operatorname{sgn}(\Gamma) \int_{\xi_A}^{\xi_B} \left[\theta_g + \xi \frac{d\theta_g}{d\xi}\right] d\xi.$$

Noting that the geometric phase changes sign with Γ in all three problems, we observe that $\triangle L$ is independent of the sign of Γ. Therefore, without loss of generality, we assume that $\operatorname{sgn}(\Gamma) = +1$ and are left with the following simple expression for $\triangle L$:

$$\triangle L = -\int_{\xi_A}^{\xi_B} d(\xi\theta_g). \tag{15}$$

Thus, for an interface of arbitrary shape and orientation, the geometric phase causes the above additional $O(1)$ term to appear in the length evolution. For an initially linear interface coincident with a ray from the point vortex, θ_i and hence θ_g are independent of ξ, and the $O(1)$ contribution takes on the simpler form,

$$\triangle L = -\theta_g \int_{\xi_A}^{\xi_B} d\xi = -\theta_g(\xi_B - \xi_A).$$

Circumferential Interfaces. We now analyse the change in length for an interface that initially coincides with a portion of a circular streamline of the parent vortex. We assume as before that the interface connects two points labelled A and B. ξ now assumes a constant value Π along the interface. The parametrization with respect to ξ breaks down and we therefore parametrize the interface with respect to θ_i. The length integral (4) assumes the form,

$$L_\epsilon(t) = \int_{(\theta_i)_A}^{(\theta_i)_B} \sqrt{\left(r\frac{d\theta}{d\theta_i}\right)^2 + \left(\frac{dr}{d\theta_i}\right)^2} d\theta_i,$$

where $(\theta_i)_A \le \theta_i \le (\theta_i)_B$. It is trivial to see that in the absence of the background field, the interface length remains constant ($= L_i$, say) for all times. We evaluate this integral in the presence of the background field using (10) and (11) with $\xi = \Pi$. It is straightforward to see that

$$\begin{aligned} L_\epsilon(T) &= \int_{(\theta_i)_A}^{(\theta_i)_B} \left[\Pi \frac{d}{d\theta_i}\left(\theta_i + \theta_g\right) + O\left(\frac{1}{\sqrt{T}}\right)\right] d\theta_i \\ &= L_i + \int_{(\theta_i)_A}^{(\theta_i)_B} \left[\Pi \frac{d\theta_g}{d\theta_i} + O\left(\frac{1}{\sqrt{T}}\right)\right] d\theta_i. \end{aligned}$$

Hence,

$$\triangle L = \int_{(\theta_i)_A}^{(\theta_i)_B} \Pi \frac{d\theta_g}{d\theta_i} d\theta_i = \int_{(\theta_i)_A}^{(\theta_i)_B} d\left(\xi\theta_g\right) = -\int_{(\theta_i)_B}^{(\theta_i)_A} d\left(\xi\theta_g\right),$$

[4] Justifying this assumption would, of course, require a more detailed and careful analysis.

which is an integral formula exactly like (15) except that here the upper limit of the integral is the lower bound of the parameter. To calculate the length change for an interface with both transversal and circumferential portions (such as, for example, closed loops), one partitions the interface into these portions and applies (15) in each portion appropriately. The length change for the whole interface being the sum of the length changes over all portions.

Remarks on the Derivation of (15).

1. The nondimensional asymptotic solutions for (r, θ) as given by (7), (8) were derived in Part I, and we do not repeat the details here. The same is true for each of the model flows in the next section.
2. The expansions (10), (11) and (12), (13) are viewed as formal in the sense that we have made assumptions about the boundedness of the higher order terms and their derivatives with respect to ξ.
3. In Step 3, use is made of the Riemann-Lebsegue lemma, which says that if a real-valued function $f(x)$ is Riemann-integrable on the real interval (a, b), then

$$\lim_{\lambda\to\infty}\int_a^b f(x)e^{i\lambda x}\,dx = 0, \qquad \lambda \in \mathbf{R}.$$

   For a precise statement of the lemma, see Sirovich [14].

We now give details of these calculations in each of the problems. We split the calculations into the three steps as outlined in this section. We assume that in all problems the vortex strengths are positive.

4. Model Flows

4.1. Two Vortices with an Interface

We perform the interface calculation first in a flow in the unbounded plane due to two like-signed vortices of strengths Γ_1 and Γ_3 separated by a distance D. The vortices rotate uniformly with constant D about the fixed center of vorticity of the configuration. As mentioned in Section 2, the geometric phase for a particle close to Γ_1 in this flow is a special case of the geometric phase for the three-vortex problem obtained in Part I and is given by $\theta_g = 2\pi\Gamma_3\cos 2\theta_i/(\Gamma_1+\Gamma_3)$.

Step 1. The asymptotic solutions for the particle motion are

$$\begin{aligned} r(t,\tau,\theta_i) &= 1+\epsilon^2\left[\frac{\alpha_3}{\alpha_1}\left(\frac{\cos(2\beta(t,\tau))}{2}-\frac{\cos 2\theta_i}{2}\right)\right]+O(\epsilon^3),\\ \theta(t,\tau,\theta_i) &= \frac{\alpha_1}{2\pi}t+\frac{\alpha_3}{2\pi}\tau\cos 2\theta_i+\theta_i+\epsilon\tilde{\theta}_1(\tau)\\ &\quad +\epsilon^2\left[\frac{\alpha_3}{\alpha_1}\sin(2\beta(t,\tau))+\tilde{\theta}_2(\tau)\right]+O(\epsilon^3), \end{aligned}$$

where

$$\beta(t,\tau,\theta_i) = [\alpha_1 + \alpha_3 (1-\cos 2\theta_i)]\,\frac{\tau}{2\pi} - \frac{\alpha_1}{2\pi}t - \theta_i .$$

r and t are defined as in (9), and

$$\epsilon = \frac{\xi}{D}, \qquad \alpha_j = \frac{\Gamma_j}{\omega\xi^2}, \qquad (j=1,3).$$

Step 2. The (dimensional) time period of the two vortices is given by $T = 4\pi^2 D^2/(\Gamma_1+\Gamma_3) = cD^2$. Converting all quantities to their dimensional form, the above series at the end of T becomes

$$\begin{aligned}
\hat{r}(T,\xi) &= \xi + \frac{c\Gamma_3}{\Gamma_1}\frac{\xi^3}{T}\left[\frac{1}{2}\cos\left(2\beta\left(\frac{T}{\xi^2},\xi\right)\right) - \frac{1}{2}\cos 2\theta_i\right] + O\left(\frac{1}{T\sqrt{T}}\right),\\
\theta(T,\xi) &= \frac{\Gamma_1}{2\pi}\frac{T}{\xi^2} + \theta_g + \theta_i + \frac{\sqrt{c}\xi}{\sqrt{T}}\theta_1(\xi)\\
&\quad + \frac{c\xi^2}{T}\left[\frac{\Gamma_3}{\Gamma_1}\sin\left(2\beta\left(\frac{T}{\xi^2},\xi\right)\right) + \tilde{\theta}_2(\xi)\right] + O\left(\frac{1}{T\sqrt{T}}\right),
\end{aligned}$$

where

$$\begin{aligned}
\beta\left(\frac{T}{\xi^2},\xi\right) &= [\Gamma_1 + \Gamma_3(1-\cos 2\theta_i)]\,\frac{2\pi}{\Gamma_1+\Gamma_3} - \frac{\Gamma_1}{2\pi}\frac{T}{\xi^2} - \theta_i\\
&= 2\pi - \frac{\Gamma_1}{2\pi}\frac{T}{\xi^2} - \left(\theta_g + \theta_i\right),
\end{aligned}$$

and θ_i in general can be a function of ξ. Differentiating with respect to ξ, we get the series

$$\begin{aligned}
\frac{d\hat{r}}{d\xi}(T,\xi) &= 1 - \frac{c\Gamma_3}{\pi}\sin 2\beta + O\left(\frac{1}{\sqrt{T}}\right),\\
\frac{d\theta}{d\xi}(T,\xi) &= -\frac{\Gamma_1}{\pi\xi^3}T + \frac{d}{d\xi}\left(\theta_g + \theta_i\right) + \frac{2c\Gamma_3}{\pi\xi}\cos 2\beta + O\left(\frac{1}{\sqrt{T}}\right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\left[\hat{r}\frac{d\theta}{d\xi}\right]^2 &= \left[-\frac{\Gamma_1}{\pi\xi^2}T + E + O\left(\frac{1}{\sqrt{T}}\right)\right]^2\\
&= \left\{\frac{\Gamma_1^2}{\pi^2\xi^4}\right\}T^2 - \left\{\frac{2E\Gamma_1}{\pi\xi^2}\right\}T + O(\sqrt{T}),\\
\left[\frac{d\hat{r}}{d\xi}\right]^2 &= O(1),
\end{aligned}$$

where $E = (c\Gamma_3/\pi)(3\cos 2\beta/2 + \cos 2\theta_i/2) + \xi(d/d\xi)(\theta_g + \theta_i)$. Hence,

$$\begin{aligned}\sqrt{\left[\hat{r}\frac{d\theta}{d\xi}\right]^2 + \left[\frac{d\hat{r}}{d\xi}\right]^2} &= \sqrt{\left\{\frac{\Gamma_1^2}{\pi^2\xi^4}\right\} T^2 \left[1 - \left\{\frac{2E}{\Gamma_1/\pi\xi^2}\right\}\frac{1}{T} + \cdots\right]} \\ &= \left|\frac{\Gamma_1 T}{\pi\xi^2}\right| \left[1 - \left\{\frac{E}{\Gamma_1/\pi\xi^2}\right\}\frac{1}{T} + \cdots\right] \\ &= \frac{\Gamma_1 T}{\pi\xi^2} - E + O\left(\frac{1}{\sqrt{T}}\right).\end{aligned}$$

Using (2) and (4) and writing out E, we then get

$$\triangle L(T) = -\int_{\xi_A}^{\xi_B} \left[\frac{c\Gamma_3}{\pi}\left(\frac{3}{2}\cos 2\beta + \frac{\cos 2\theta_i}{2}\right) + \xi\frac{d\theta_g}{d\xi} + O\left(\frac{1}{\sqrt{T}}\right)\right] d\xi. \tag{16}$$

Step 3. The second and third terms in the integrand depend on ξ alone and will therefore give $O(1)$ terms after integration. We now examine the order of the first term after integration:

$$\begin{aligned}\int_{\xi_A}^{\xi_B} \cos 2\beta d\xi &= \int_{\xi_A}^{\xi_B} \cos\left(2\left\{2\pi - \frac{\Gamma_1}{2\pi}\frac{T}{\xi^2} - \left(\theta_g + \theta_i\right)\right\}\right) d\xi \\ &= \int_{\xi_A}^{\xi_B} \cos\left(\frac{\Gamma_1}{\pi}\frac{T}{\xi^2} + 2\left(\theta_g + \theta_i\right)\right) d\xi \\ &= \int_{\xi_A}^{\xi_B} \cos\left(\frac{\Gamma_1}{\pi}\frac{T}{\xi^2}\right)\cos 2\left(\theta_g + \theta_i\right) d\xi \\ &\quad - \int_{\xi_A}^{\xi_B} \sin\left(\frac{\Gamma_1}{\pi}\frac{T}{\xi^2}\right)\sin 2\left(\theta_g + \theta_i\right) d\xi.\end{aligned}$$

By the Riemann-Lebesgue lemma, each of the integrals in the last line vanishes in the limit $T \to \infty$. Making the further assumption that all other (higher order) terms in the integrand of (16) are of the same form and hence have vanishing limits leads us to the result,

$$\begin{aligned}\triangle L = \lim_{T\to\infty} \triangle L(T) &= -\int_{\xi_A}^{\xi_B} \left(\xi\frac{d\theta_g}{d\xi} + \frac{c\Gamma_3\cos 2\theta_i}{2\pi}\right) d\xi \\ &= -\int_{\xi_A}^{\xi_B} \left(\xi\frac{d\theta_g}{d\xi} + \theta_g\right) d\xi \\ &= -\int_{\xi_A}^{\xi_B} d\left(\xi\theta_g\right).\end{aligned}$$

In terms of the initial angle θ_i and the vortex strengths, the result is

$$\triangle L = \frac{4\pi\Gamma_3}{\Gamma_1 + \Gamma_3}\int_{\xi_A}^{\xi_B} \left(\xi\sin 2\theta_i\frac{d\theta_i}{d\xi} - \frac{\cos 2\theta_i}{2}\right) d\xi.$$

In the particular case of an interface initially coincident with a ray from the parent vortex (Γ_1), we get

$$\begin{aligned}\triangle L &= -\theta_g \int_{\xi_A}^{\xi_B} d\xi \\ &= -\theta_g(\xi_B - \xi_A) \\ &= -\frac{2\pi\Gamma_3}{\Gamma_1 + \Gamma_3} \cos 2\theta_i (\xi_B - \xi_A).\end{aligned}$$

The result for this case can also be found in Newton and Shashikanth [15].

4.2. Vortex and Interface in a Circular Domain

We next perform the interface computation in the flow due to a point vortex in a circular domain (in the plane) of radius R_2. The point vortex moves with constant speed in a circular orbit of radius R_1 ($0 < R_1 < R_2$). The geometric phase for a fluid particle closer to the point vortex than to the circular boundary was calculated in Part I and is given by $\theta_g = -(2\pi/b)\cos 2\theta_i$, where $b = (R_2/R_1)^2 - 1$.

Step 1. The asymptotic solutions for the particle motion are

$$\begin{aligned}r(t,\tau,\theta_i) &= 1 + \epsilon^2\left[-\frac{1}{2}\cos(2\beta(t,\tau)) + \frac{1}{2}\cos 2\theta_i\right] + O(\epsilon^3), \\ \theta(t,\tau,\theta_i) &= \frac{\alpha}{2\pi}(t - \tau\cos 2\theta_i) + \theta_i + \epsilon\tilde{\theta}_1(\tau) \\ &\quad + \epsilon^2\left[-\sin(2\beta(t,\tau)) + \tilde{\theta}_2(\tau)\right] + O(\epsilon^3),\end{aligned}$$

where

$$\beta(t,\tau,\theta_i) = \frac{\alpha}{2\pi}\left[(b + \cos 2\theta_i)\tau - t\right] - \theta_i.$$

It can easily be shown that $D = bR_1$ is the distance between the point vortex and its image vortex in an equivalent unbounded flow. r and t are defined as in (9), and

$$\epsilon = \frac{\xi}{bR_1} = \frac{\xi}{D}, \qquad \alpha = \frac{\Gamma}{\omega\xi^2}.$$

Step 2. The (dimensional) time period is given by $T = 4\pi^2 bR_1^2/\Gamma = 4\pi^2 D^2/b\Gamma = cD^2$. In dimensional form, the above series at the end of T are

$$\begin{aligned}\hat{r}(T,\xi) &= \xi + \frac{c\xi^3}{T}\left[-\frac{1}{2}\cos\left(2\beta\left(\frac{T}{\xi^2},\xi\right)\right) + \frac{1}{2}\cos 2\theta_i\right] + O\left(\frac{1}{T\sqrt{T}}\right), \\ \theta(T,\xi) &= \frac{\Gamma}{2\pi}\frac{T}{\xi^2} + \theta_g + \theta_i + \frac{\sqrt{c}\xi}{\sqrt{T}}\theta_1(\xi) \\ &\quad + \frac{c\xi^2}{T}\left[-\sin\left(2\beta\left(\frac{T}{\xi^2},\xi\right)\right) + \tilde{\theta}_2(\xi)\right] + O\left(\frac{1}{T\sqrt{T}}\right),\end{aligned}$$

where

$$\begin{aligned}\beta\left(\frac{T}{\xi^2},\xi\right) &= 2\pi\left(1+\frac{\cos 2\theta_i}{b}\right)-\frac{\Gamma}{2\pi}\frac{T}{\xi^2}-\theta_i\\ &= 2\pi-\frac{\Gamma}{2\pi}\frac{T}{\xi^2}-\left(\theta_g+\theta_i\right),\end{aligned}$$

and θ_i in general can be a function of ξ. Differentiating with respect to ξ, we get the series

$$\begin{aligned}\frac{d\hat{r}}{d\xi}(T,\xi) &= 1+\frac{c\Gamma}{\pi}\sin 2\beta+O\left(\frac{1}{\sqrt{T}}\right),\\ \frac{d\theta}{d\xi}(T,\xi) &= -\frac{\Gamma}{\pi\xi^3}T+\frac{d}{d\xi}\left(\theta_g+\theta_i\right)-\frac{2c\Gamma}{\pi\xi}\cos 2\beta+O\left(\frac{1}{\sqrt{T}}\right).\end{aligned}$$

Therefore,

$$\begin{aligned}\left[\hat{r}\frac{d\theta}{d\xi}\right]^2 &= \left[-\frac{\Gamma}{\pi\xi^2}T+E+O\left(\frac{1}{\sqrt{T}}\right)\right]^2\\ &= \left\{\frac{\Gamma^2}{\pi^2\xi^4}\right\}T^2-\left\{\frac{2E\Gamma}{\pi\xi^2}\right\}T+O(\sqrt{T}),\\ \left[\frac{d\hat{r}}{d\xi}\right]^2 &= O(1),\end{aligned}$$

where $E=(c\Gamma/\pi)\,(-3\cos 2\beta/2-\cos 2\theta_i/2)+\xi(d/d\xi)\left(\theta_g+\theta_i\right)$. Hence,

$$\begin{aligned}\sqrt{\left[\hat{r}\frac{d\theta}{d\xi}\right]^2+\left[\frac{d\hat{r}}{d\xi}\right]^2} &= \sqrt{\left\{\frac{\Gamma^2}{\pi^2\xi^4}\right\}T^2\left[1-\left\{\frac{2E}{\Gamma/\pi\xi^2}\right\}\frac{1}{T}+O\left(\frac{1}{T\sqrt{T}}\right)\right]}\\ &= \left|\frac{\Gamma T}{\pi\xi^2}\right|\left[1-\left\{\frac{E}{\Gamma/\pi\xi^2}\right\}\frac{1}{T}+O\left(\frac{1}{T\sqrt{T}}\right)\right]\\ &= \frac{\Gamma T}{\pi\xi^2}-E+O\left(\frac{1}{\sqrt{T}}\right).\end{aligned}$$

Using (2) and (4) and writing out E, we then get

$$\triangle L(T) = \int_{\xi_A}^{\xi_B}\left[\frac{c\Gamma}{\pi}\left(\frac{3}{2}\cos 2\beta+\frac{\cos 2\theta_i}{2}\right)-\xi\frac{d\theta_g}{d\xi}+O\left(\frac{1}{\sqrt{T}}\right)\right]d\xi. \qquad (17)$$

Step 3. As in the previous problem, the integral of the first term has a zero limit as $T\to\infty$. Making the further assumption that all other (higher order) terms in the integrand of (17) have a similiar form, and hence have vanishing limits, leads us to the result,

$$\triangle L=\lim_{T\to\infty}\triangle L(T) = -\int_{\xi_A}^{\xi_B}\left(\xi\frac{d\theta_g}{d\xi}-\frac{c\Gamma\cos 2\theta_i}{2\pi}\right)d\xi$$

$$= -\int_{\xi_A}^{\xi_B} \left(\xi \frac{d\theta_g}{d\xi} + \theta_g \right) d\xi$$

$$= -\int_{\xi_A}^{\xi_B} d\left(\xi\theta_g\right).$$

Since the geometric phase in this problem is independent of the vortex strength, so is $\triangle L$. In terms of the initial angle θ_i, the result is

$$\triangle L = \frac{4\pi}{b} \int_{\xi_A}^{\xi_B} \left(-\xi \sin 2\theta_i \frac{d\theta_i}{d\xi} + \frac{\cos 2\theta_i}{2} \right) d\xi.$$

In the particular case of an interface initially coincident with a ray from the parent vortex, we get

$$\begin{aligned} \triangle L &= -\theta_g \int_{\xi_A}^{\xi_B} d\xi \\ &= -\theta_g(\xi_B - \xi_A) \\ &= \frac{2\pi}{b} \cos 2\theta_i (\xi_B - \xi_A). \end{aligned}$$

5. Mixing Layer Model

In our final problem, we perform the interface computation for the mixing layer model of Part I. As illustrated in Figure 4, vortex-pairing is induced in this configuration of initially stationary vortices by a subharmonic perturbation. The geometric phase for a fluid particle in this point-vortex flow was calculated in Part I and is given by $\theta_g = [(2k+6)/3]K \cos 2\theta_i$ (k and K are defined below).

Step 1. The asymptotic solutions for the particle motion are

$$\begin{aligned} r(t, \tau, \theta_i) &= 1 + \epsilon^2 \left[\frac{\tilde{S}_h(\tau)\tilde{S}_t(\tau)}{4L^2} \sin 2\theta_0 + \left(\frac{2}{3} - \frac{\tilde{S}_h^2(\tau) - \tilde{S}_t^2(\tau)}{2L^2} \right) \cos 2\theta_0 \right. \\ &\qquad \left. - \left(\frac{k+3}{24k} \right) \cos 2\theta_i \right] + O(\epsilon^3), \\ \theta(t, \tau, \theta_i) &= \theta_0(t, \tau, \theta_i) + \epsilon\tilde{\theta}_1(\tau) + \epsilon^2 \left[\frac{\tilde{S}_h(\tau)\tilde{S}_t(\tau)}{2L^2} \cos 2\theta_0 \right. \\ &\qquad \left. + \left(\frac{\tilde{S}_h^2(\tau) - \tilde{S}_t^2(\tau)}{4L^2} - \frac{1}{3} \right) \sin 2\theta_0 + \tilde{\theta}_2(\tau) \right] \\ &\qquad + O(\epsilon^3), \end{aligned}$$

where $\theta_0(t, \tau, \theta_i) = 2kt + \left(\frac{k+3}{6}\right) \tau \cos 2\theta_i + \theta_i$ and $\tilde{S}_h, \tilde{S}_t$ are functions of the vortex motion and hence vary on the slow time alone. L is an invariant of the vortex motion.

These terms are therefore independent of both T/ξ^2 and ξ and do not play a role in our derivation. To simplify notation, we define $F = \tilde{S}_h\tilde{S}_t/4L^2$ and $G = (\tilde{S}_h^2 - \tilde{S}_t^2)/(4L^2) - 1/3$. r and t are defined as in (9), and

$$\epsilon = \delta\frac{\xi}{D_i}, \qquad k = \frac{\Gamma}{4\pi\omega\xi^2},$$

where $\delta = \pi D_i/a$, a is the intervortex spacing for the unperturbed configuration and D_i is the initial intervortex spacing for the perturbed configuration (see Part I for details). k (which depends on δ) is the modulus of the elliptic integrals that appear in the vortex solutions.

Step 2. The (dimensional) time period of the vortex motion is given by

$$T = (16kK/\pi\Gamma)a^2 = (16\pi kK/\Gamma\delta^2)D_i^2 = a_2 D_i^2,$$

where $K(k)$ is the complete elliptic integral of the first kind. Defining $c = \delta^2 a_2$, we write the above series in dimensional form at the end of time T,

$$\begin{aligned}
\hat{r}(T,\xi) &= \xi + \frac{c\xi^3}{T}\left[F\sin 2\theta_0 - 2G\cos 2\theta_0 - \left(\frac{k+3}{24k}\right)\cos 2\theta_i\right] + O\left(\frac{1}{T\sqrt{T}}\right),\\
\hat{\theta}(T,\xi) &= \frac{\Gamma}{2\pi}\frac{T}{\xi^2} + \theta_g + \theta_i + \frac{\sqrt{c}\xi}{\sqrt{T}}\theta_1(\xi)\\
&\quad + \frac{c\xi^2}{T}\left[2F\cos 2\theta_0 + G\sin 2\theta_0 + \tilde{\theta}_2\right] + O\left(\frac{1}{T\sqrt{T}}\right),
\end{aligned}$$

where

$$\theta_0\left(\frac{T}{\xi^2},\xi\right) = \frac{\Gamma}{2\pi}\frac{T}{\xi^2} + \theta_g + \theta_i.$$

Differentiating with respect to ξ, we get the series

$$\begin{aligned}
\frac{d\hat{r}}{d\xi}(T,\xi) &= 1 + \frac{2c\Gamma}{\pi}\left[-F\cos 2\theta_0 - 2G\sin 2\theta_0\right] + O\left(\frac{1}{\sqrt{T}}\right),\\
\frac{d\theta}{d\xi}(T,\xi) &= -\frac{\Gamma}{\pi\xi^3}T + \frac{d}{d\xi}\left(\theta_g + \theta_i\right)\\
&\quad + \frac{2c\Gamma}{\pi\xi}\left[G\cos 2\theta_0 - 2F\sin 2\theta_0\right] + O\left(\frac{1}{\sqrt{T}}\right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\left[\hat{r}\frac{d\theta}{d\xi}\right]^2 &= \left\{-\frac{\Gamma}{\pi\xi^2}T + E + O\left(\frac{1}{\sqrt{T}}\right)\right\}^2\\
&= \left\{\frac{\Gamma^2}{\pi^2\xi^4}\right\}T^2 - \left\{\frac{2E\Gamma}{\pi\xi^2}\right\}T + O(\sqrt{T}),\\
\left[\frac{d\hat{r}}{d\xi}\right]^2 &= O(1),
\end{aligned}$$

where

$$E = -(c\Gamma/\pi)\,[5F\sin 2\theta_0 - 4G\cos 2\theta_0 - (k+3)\cos 2\theta_i/(24k)] + \xi(d/d\xi)(\theta_g + \theta_i).$$

Hence,

$$\begin{aligned}\sqrt{\left[\hat r\frac{d\theta}{d\xi}\right]^2 + \left[\frac{d\hat r}{d\xi}\right]^2} &= \sqrt{\left\{\frac{\Gamma^2}{\pi^2\xi^4}\right\}T^2\left[1 - \left\{\frac{2E}{\Gamma/\pi\xi^2}\right\}\frac{1}{T} + O\left(\frac{1}{T\sqrt{T}}\right)\right]} \\ &= \left|\frac{\Gamma T}{\pi\xi^2}\right|\left[1 - \left\{\frac{E}{\Gamma/\pi\xi^2}\right\}\frac{1}{T} + O\left(\frac{1}{T\sqrt{T}}\right)\right] \\ &= \frac{\Gamma T}{\pi\xi^2} - E + O\left(\frac{1}{\sqrt{T}}\right).\end{aligned}$$

Using (2) and (4) and writing out E, we then get

$$\begin{aligned}\triangle L(T) = \int_{\xi_A}^{\xi_B}\Bigg\{\frac{c\Gamma}{\pi}&\left[5F\sin 2\theta_0 - 4G\cos 2\theta_0 - \left(\frac{k+3}{24k}\right)\cos 2\theta_i\right] \\ &- \xi\frac{d\theta_g}{d\xi} + O\left(\frac{1}{\sqrt{T}}\right)\Bigg\}\,d\xi. \qquad (18)\end{aligned}$$

Step 3. As in the previous problems, it is easy to show that the integrals of the first and second terms vanish in the limit $T \to \infty$. Making our assumption that all other (higher order) terms in the integrand of (18) have a similiar form and hence vanishing limits leads us to the result,

$$\begin{aligned}\triangle L = \lim_{T\to\infty}\triangle L(T) &= -\int_{\xi_A}^{\xi_B}\left[\xi\frac{d\theta_g}{d\xi} + \frac{c\Gamma}{\pi}\left(\frac{k+3}{24k}\right)\cos 2\theta_i\right]d\xi \\ &= -\int_{\xi_A}^{\xi_B}\left[\xi\frac{d\theta_g}{d\xi} + \theta_g\right]d\xi \\ &= -\int_{\xi_A}^{\xi_B} d(\xi\theta_g).\end{aligned}$$

As in the previous problem, since θ_g is independent of the vortex strength so is $\triangle L$. In terms of the initial angle θ_i, the result is

$$\triangle L = \int_{\xi_A}^{\xi_B}\left[-\left(\frac{2k+6}{3}\right)K\cos 2\theta_i + \left(\frac{4k+12}{3}\right)K\xi\sin 2\theta_i\frac{d\theta_i}{d\xi}\right]d\xi.$$

For an interface initially coincident with a ray from the parent vortex, we get

$$\begin{aligned}\triangle L &= -\theta_g\int_{\xi_A}^{\xi_B} d\xi \\ &= -\theta_g(\xi_B - \xi_A) \\ &= -K\left(\frac{2k+6}{3}\right)\cos 2\theta_i(\xi_B - \xi_A).\end{aligned}$$

6. Conclusion

We show in this paper that the geometric phase exhibited in the angle variable of a passive particle in the three 'canonical' vortex configurations considered in Part I also affects the evolution of an interface of passive particles in these flows. The interface wraps into a spiral structure around the parent vortex with a slowly varying component induced by the farfield vortices. An extra term appears in the length of the interface over long time periods that depends on the geometric phase and, like the phase, is also geometric. It is the integral over the initial interface of a perfect differential, or an exact form, of a function that is a weighted geometric phase term. We believe that from the fluid dynamics point of view this result, based on the results of Part I, is a further step towards understanding the significance and implications of the geometric phase in fluid flows. From a more practical point of view, the results in this paper may be useful as theoretical estimates allowing for future quantitative comparisons with numerical and experimental work.

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